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On Cartan's Conformally Deformable Hypersurfaces

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Starting in 1916, E. Cartan devoted five years to the study of isometric, conformal, and projective deformations of submanifolds by the use of the method of moving frames. In the first of a series of papers ([2]; see also [9]), he locally classified the hypersurfaces M^n ($n \geq 3$) in flat Euclidean space $\mathbb{R}^{n+1}$ that are isometrically deformable. Shortly after, he followed with a long and more difficult paper [3] where he classified conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$. The special cases $n = 4, 3$ were subsequently treated by Cartan in [4] and [5]. In all cases, it turns out that hypersurfaces are generically conformally rigid.

Quite similarly to the isometric case, conformally deformable hypersurfaces of dimension $n \geq 5$, other than the conformally flat ones, can be separated into four classes: surface-like, conformally ruled, those having precisely a continuous 1-parameter family of deformations, and those that admit only one deformation. Cartan's main result is a parametric description of the hypersurfaces in the last two classes as envelopes of 2-parameter families of spheres determined by a certain partial differential equation together with an additional condition.

Our first and main achievement is a nonparametric classification of all conformally deformable Euclidean hypersurfaces of dimension $n \geq 5$ by means of a rather simple geometric construction. Roughly speaking, we show that any hypersurface M^n in $\mathbb{R}^{n+1}$ ($n \geq 5$) that admits a conformal deformation $\tilde{M}^n$ can be locally characterized as the intersection $M^n = N^{n+1} \cap \mathbb{V}$ of a flat $(n+1)$-dimensional Riemannian submanifold of the standard flat Lorentzian space $\mathbb{L}^{n+3}$ with the light cone $\mathbb{V}$ of $\mathbb{L}^{n+3}$. Moreover, $\tilde{M}^n$ is obtained by projecting M^n onto the standard model of $\mathbb{R}^{n+1}$ as an embedded hypersurface of $\mathbb{V}$. In addition, we characterize how the conformally deformable hypersurfaces that are conformally congruent to isometrically deformable ones can be produced by the procedure just described. They are the ones obtained from the flat Riemannian submanifolds whose relative nullity leaves are open subsets of affine subspaces in $\mathbb{L}^{n+3}$ with a common point in $\mathbb{V}$.

For reasons we can only guess (perhaps uncertainty about the very existence of examples), Cartan's statement in the introduction of [3] completely ignores the discrete last class (although the possibility of its existence arises in his proof; see Sec. 41). This raises the question of whether the discrete class is nonempty. The

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