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On Elliptic $K3$ Surfaces

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1. Introduction

By virtue of Torelli's theorem for the period map on the moduli of complex $K3$ surfaces [4; 13; 18], we can study many aspects of $K3$ surfaces from the lattice-theoretic point of view. In this paper, we determine all possible ADE-types of singular fibers of elliptic $K3$ surfaces using Nikulin's theory of discriminant forms of even integral lattices. We also determine, for each ADE-type of singular fibers, all possible torsion parts of the Mordell–Weil groups. Throughout this paper, we use the term "an elliptic $K3$ surface" for "a complex elliptic $K3$ surface with a distinguished zero section" and the term "an elliptic fibration" for "a complex Jacobian elliptic fibration".

A finite formal sum of the symbols A_l $(l \geq 1)$, D_m $(m \geq 4)$, and E_n $(n = 6, 7, 8)$ with nonnegative integer coefficients is called an *ADE-type*. For an ADE-type

$$\Sigma := \sum a_l A_l + \sum d_m D_m + \sum e_n E_n,$$

we denote by $L(\Sigma)^-$ the negative definite root lattice generated by a root system of type Σ, and by $\operatorname{rank}(\Sigma)$ the rank of $L(\Sigma)^-$. By definition, we have $\operatorname{rank}(\Sigma) = \sum a_l l + \sum d_m m + \sum e_n n$.

Let $f\colon X \to \mathbb{P}^1$ be an elliptic $K3$ surface, and let $O\colon \mathbb{P}^1 \to X$ be the zero section of f. Let MW_f be the Mordell–Weil group of f. The torsion part of MW_f is a finite abelian group, which we shall denote by G_f. We put

$$R_f := \{\, p \in \mathbb{P}^1 \mid f^{-1}(p) \text{ is reducible} \,\}$$

and, for each $p \in R_f$, we denote by $f^{-1}(p)^\sharp$ the union of irreducible components of $f^{-1}(p)$ that are disjoint from the zero section. It is known that the cohomology classes of irreducible components of $f^{-1}(p)^\sharp$ span a negative definite root lattice generated by an indecomposable root system of type A_l, D_m, or E_n. Let $\tau_{f,p}$ be the type. The type of singular fiber $f^{-1}(p)$ in the list of Kodaira's classification [7] is related to $\tau_{f,p}$ in an almost one-to-one way (cf. Table 2.8). We define the ADE-type Σ_f of $f\colon X \to \mathbb{P}^1$ by

$$\Sigma_f := \sum_{p \in R_f} \tau_{f,p}.$$

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