Michigan Math. J. 47 (2000)

Boundary Values and Mapping Degree

Edgar Lee Stout

Introduction

This note is an addendum to the paper of Alexander and Wermer [2], in which the authors relate the theory of linking numbers to the question of finding an analytic variety bounded by a given real, odd-dimensional submanifold of $\mathbb{C}^N$.

We give a characterization of the boundary values of holomorphic functions on certain domains in $\mathbb{C}^N$ in similar terms. In fact, the work of Alexander and Wermer contains such a characterization *in the case of functions of class* $\mathscr{C}^1$. It seems that the methods used in [2] require this degree of smoothness, but we have found that it is possible to obtain a result that characterizes the *continuous* functions that are boundary values of holomorphic functions that is entirely in the spirit of [2]. Specifically, we shall prove the following result.

Main Theorem. *Let Ω be a bounded domain in $\mathbb{C}^N$ with boundary of class $\mathscr{C}^2$, and assume that $\bar{\Omega}$ has a Stein neighborhood basis. A continuous function f on $b\Omega$ is of the form $F\big|_{b\Omega}$ for a function $F \in \mathscr{C}(\bar{\Omega})$ that is holomorphic on Ω if and only if the following condition is met.*

(∗) *With Γ_f the graph $\{(z, f(z)) : z \in b\Omega\}$, a compact subset of $\mathbb{C}^{N+1}$, if Q is a $\mathbb{C}^N$-valued holomorphic map defined on a neighborhood of $\bar{\Omega} \times \mathbb{C}$ with $Q^{-1}(0) \cap \Gamma_f = \emptyset$, then the degree of the map $b\Omega \to \mathbb{C}^N \setminus \{0\}$ given by $z \mapsto Q(z, f(z))$ is nonnegative.*

Recall that a closed set E in $\mathbb{C}^N$ is said to have a Stein neighborhood basis if it is the intersection of a sequence of domains of holomorphy in $\mathbb{C}^N$. If E is the closure of a strictly pseudoconvex domain or a polydisc in $\mathbb{C}^N$, then it has a Stein neighborhood basis.

The case of the main theorem in which f is of class $\mathscr{C}^1$ is contained in [2] as a very special case of the main results of that paper.

The main theorem seems to be new, even in the setting of classical function theory, where a version of the result is the following. Let $\mathbb{U}$ denote the open unit disc in the complex plane.

Corollary. *A continuous function f on $b\mathbb{U}$ extends holomorphically through $\mathbb{U}$ if and only if, for each polynomial $p(z) = p(z_1, z_2)$ in two complex variables*

Received November 12, 1999.