Green's Functions for Irregular Quadratic Polynomial Automorphisms of $\mathbb{C}^3$

Dan Coman & John Erik Fornæss

1. Introduction

We first recall some basic facts about the dynamics of (holomorphic) polynomial automorphisms of $\mathbb{C}^2$. It was shown in [FM] that the ones with interesting dynamics are affinely conjugated to a finite composition of generalized Hénon maps—that is, maps of the form $(x, y) \to (P(x) - ay, x)$, where P is a holomorphic polynomial in $\mathbb{C}$. The dynamics of such maps is studied in detail in a sequence of papers by Bedford and Smillie, starting with [BS], in [FS], and also in [H].

For simplicity, let us refer to the case of one generalized Hénon map, $h(x, y) = (P(x) - ay, x)$, where P has degree $d \geq 2$. There are two posibilities: either the forward iterates h^n of h can escape to infinity at super-exponential rate ($\sim$ (const)$^{d^n}$) or they are locally uniformly bounded. The first situation occurs on an open set U^+ and the second on the complement $K^+ = \mathbb{C}^2 \setminus U^+$. Then the Fatou set of h, defined in the usual sense as the largest open set on which the iterates $\{h^n\}$ form a normal family, is given by $U^+ \cap \operatorname{int} K^+$, while the Julia set is ∂K^+. Similar statements hold for the inverse map h^{-1}, the corresponding sets being denoted by U^- and K^-. Using these facts, one defines (pluricomplex) Green's functions which measure the (super-exponential) rate of escape to infinity in forward/backward time:

$$G^{\pm}(w) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \|h^{\pm n}(w)\|,$$

where $w = (x, y) \in \mathbb{C}^2$. These functions are continuous plurisubharmonic on $\mathbb{C}^2$ and actually pluriharmonic on U^+ (resp. on U^-). Moreover, $K^{\pm} = \{G^{\pm} = 0\}$. The Green's functions are used to define the currents $\mu^{\pm} = dd^c G^{\pm}$, supported on $\partial K^{\pm}$, which satisfy $h^{\star}\mu^{\pm} = d^{\pm 1}\mu^{\pm}$ $\big($here $d^c = \frac{1}{2\pi i}(\partial - \bar{\partial})\big)$. It follows that $\mu = \mu^+ \wedge \mu^-$ is a probability measure supported on $\partial K^+ \cap \partial K^-$, which is invariant under h. The currents $\mu^{\pm}$ and the invariant measure μ are important tools in understanding the dynamics of h.

It is an interesting problem to study the dynamics of polynomial automorphisms of $\mathbb{C}^N$ in dimensions higher than 2. To our knowledge, there are only a few attempts in this direction, which we briefly recall now. The theory of Hénon maps in $\mathbb{C}^2$ carries over to the special class of shift-like polynomial automorphisms of $\mathbb{C}^N$, which are introduced and studied in [BP].

Received September 17, 1998. Revision received May 20, 1999.
Michigan Math. J. 46 (1999).