On the Length of Lemniscates

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For a monic polynomial p of degree d, we write $E(p) := \{z : |p(z)| = 1\}$. A conjecture of Erdős, Herzog and Piranian [4], repeated by Erdős in [5, Prob. 4.10] and elsewhere, is that the length $|E(p)|$ is maximal when $p(z) := z^d + 1$. It is easy to see that, in this conjectured extremal case, $|E(p)| = 2d + O(1)$ when $d \to \infty$.

The first upper estimate $|E(p)| \le 74d^2$ is due to Pommerenke [10]. Recently, Borwein [2] gave an estimate that is linear in d, namely

$$|E(p)| \le 8\pi ed \approx 68.32d.$$

Here we improve Borwein's result.

Let α_0 be the least upper bound of perimeters of the convex hulls of compact connected sets of logarithmic capacity 1. The precise value of α_0 is not known, but Pommerenke [8] proved the estimate $\alpha_0 < 9.173$. The conjectured value is $\alpha_0 = 3^{3/2}2^{2/3} \approx 8.24$.

Theorem 1. *For monic polynomials p of degree d, $|E(p)| \le \alpha_0 d < 9.173d$.*

A similar problem for rational functions turns out to be much easier, and can be solved completely by means of Lemma 1.

Theorem 2. *Let f be a rational function of degree d. Then the spherical length of the preimage under f of any circle C is at most d times the length of a great circle.*

This is best possible, as shown by the example of $f(z) = z^d$ and $C = \mathbf{R}$.

Remarks. Borwein notices that his method would give the estimate $4\pi d \approx 12.57d$ if one knew one of the following facts: (a) the precise estimate of the size of the exceptional set in Cartan's lemma (Lemma 3 here); or (b) for extremal polynomials, the set $E(p)$ is connected. It turns out that (b) is true (this is our Lemma 3), and in addition we can improve from 4π to 9.173 by using more precise arguments than those of Borwein.

The main property of the level sets $E(p)$ is the following.

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