Birational Maps, Positive Currents, and Dynamics

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1. Introduction

There has been a great deal of recent research in multivariable complex dynamics, most of it devoted to either polynomial diffeomorphisms of $\mathbf{C}^2$ or holomorphic maps of $\mathbf{P}^n$. Pluripotential theory plays a prominent supporting role in nearly all this work. Our concern in this paper and its predecessor [Dil] is to extend the application of pluripotential theory to study dynamics of birational maps of $\mathbf{P}^2$.

Anyone who seeks to understand the dynamics of a birational map $f_+ : \mathbf{P}^2 \to \mathbf{P}^2$ faces an immediate problem: birational maps are not generally maps. That is, except when f_+ has degree $d = 1$, there exists a finite non-empty set I^+ of points where f_+ cannot be defined continuously. In a precise sense, f_+ "blows up" each of these points of indeterminacy to an entire algebraic curve. Nevertheless, we believe that it is worthwhile to pretend as far as possible that birational maps really are diffeomorphisms.

Maintaining this pretense means (among other things) that we must generalize operations like pushforward and pullback that are natural for diffeomorphisms. Since we intend to use pluripotential theory, it is particularly important to make sense of these operations as they apply to positive currents. Already in [Dil] we observed that there are at least two reasonable ways for a birational map to act on a positive closed (1, 1) current T. In order to distinguish between these actions, we refer to them as pushforward $f_{+*}T$ and pullback f_+^*T, respectively. Intuitively speaking, the first action discounts any contribution from the indeterminacy set whereas the second (defined by pulling back a potential function) takes the fullest possible account of such contributions. Theorem 2.3 gives a precise condition for agreement between pushforward by a birational map and pullback by its inverse. Namely, one has agreement if and only if the so-called Lelong numbers of T vanish at each point in I^+.

Our first application of Theorem 2.3 is to a natural current associated with iterates of a birational map. By pulling back and rescaling the Fubini–Study Kähler form Θ, one obtains a positive closed (1, 1) current

$$\mu^+ = \lim_{n\to\infty} \frac{1}{d^n} f_+^{n*}\Theta.$$

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