Expanding Factors for Pseudo-Anosov Homeomorphisms

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1. Introduction and Definitions

Thurston classified homeomorphisms on compact surfaces up to isotopy (see [3; 5]). He showed that any homeomorphism on a compact surface may be decomposed into simpler homeomorphisms on simpler compact surfaces. These simpler homeomorphisms are either periodic or pseudo-Anosov. Here we study the dynamics of the pseudo-Anosov homeomorphisms, because they are much more complicated and much richer than those of the periodic ones. In addition, pseudo-Anosov homeomorphisms on compact surfaces can be thought of as a natural extension of the study of hyperbolic toral automorphisms on the 2-dimensional torus. Using the Markov matrix, Markov partitions of these maps allow us to make a natural association with symbolic dynamics.

In the first section, we recall the basic definitions and background theorems. The second section provides several examples of pseudo-Anosov homeomorphisms on the two-dimensional sphere. In the final section, using tools from algebraic topology, we prove the following theorem, which extends a theorem concerning hyperbolic toral automorphisms on $\mathbb{T}^2$ [14].

Theorem 3.3. *Let $f: M \to M$ be a pseudo-Anosov homeomorphism on an orientable surface of genus g with oriented unstable manifolds. Let $\mathcal{P}$ be a Markov partition for f with Markov matrix $\mathcal{A}$. If f preserves the orientation of unstable manifolds, then the eigenvalues of $f_{*1}: H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ are the same as those of $\mathcal{A}$ including multiplicity, with the possible exception of some zeros and roots of unity.*

Hence, the expanding factor λ is an eigenvalue of $f_{*1}: H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$. Similarly, if f reverses the orientation of unstable manifolds, then the eigenvalues of $f_{*1}: H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$ are the same as those of $-\mathcal{A}$ including multiplicity, with the possible exception of some zeros and roots of unity. Hence $-\lambda$ is an eigenvalue of $f_{*1}: H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R})$. As a consequence of this theorem, we have the following corollary.

Corollary 3.8. *Let λ be the expanding factor of a pseudo-Anosov homeomorphism f. If λ is the root of an irreducible quadratic equation over the rationals, then λ satisfies a quadratic of the form $x^2 + nx \pm 1$, where $n \in \mathbb{Z}$ and $|\lambda| \neq 1$.*

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