Descent for Shimura Varieties

J. S. Milne

In his Corvallis article, Langlands [L, Sec. 6] stated a conjecture that identifies the conjugate of a Shimura variety by an automorphism of $\mathbb{C}$ with the Shimura variety defined by different data, and he sketched a proof that his conjecture implies the existence of canonical models. However, as J. Wildeshaus and others have pointed out to me, it is not obvious that the descent maps defined by Langlands satisfy the continuity condition necessary for the descent to be effective. In this note, I prove that they do satisfy this condition and hence that Langlands's conjecture *does* imply the existence of canonical models—this is our only proof of the existence of these models for a general Shimura variety. The proof is quite short and elementary. I give it in Section 2 after reviewing some generalities on the descent of varieties in Section 1.

Notation and Conventions. A *variety* over a field k is a geometrically reduced scheme of finite type over $\operatorname{Spec} k$ (*not* necessarily irreducible). For a variety V over a field k and a homomorphism $\sigma\colon k \to k'$, σV is the variety over k' obtained by base change. The ring of finite adèles for $\mathbb{Q}$ is denoted by $\mathbb{A}_f$.

1. Descent of Varieties

In this section, Ω is an algebraically closed field of characteristic zero. For a field $L \subset \Omega$, $A(\Omega/L)$ denotes the group of automorphisms of Ω fixing the elements of L.

Let V be a variety over Ω, and let k be a subfield of Ω. A family $(f_\sigma)_{\sigma\in A(\Omega/k)}$ of isomorphisms $f_\sigma\colon \sigma V \to V$ will be called a *descent system* if $f_{\sigma\tau} = f_\sigma \circ \sigma f_\tau$ for all $\sigma, \tau \in A(\Omega/k)$. We say that a model $(V_0, f\colon V_{0,\Omega} \to V)$ of V over k *splits* (f_σ) if $f_\sigma = f \circ (\sigma f)^{-1}$ for all $\sigma \in A(\Omega/k)$, and that a descent system is *effective* if it is split by some model over k. The next theorem restates results of Weil [W].

Theorem 1.1. *Assume that Ω has infinite transcendence degree over k. A descent system $(f_\sigma)_{\sigma\in A(\Omega/k)}$ on a quasiprojective variety V over Ω is effective if, for some subfield L of Ω finitely generated over k, the descent system $(f_\sigma)_{\sigma\in A(\Omega/L)}$ is effective.*

Received October 8, 1998. Revision received December 2, 1998.
A small part of this research was supported by the National Science Foundation.
Michigan Math. J. 46 (1999).