Commuting Toeplitz Operators on the Harmonic Bergman Space

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1. Introduction

For $p \geq 1$, we let $L^p = L^p(D, A)$ denote the usual Lebesgue space of the open unit disk D in the complex plane. Here, the letter A denotes the normalized area measure on D. The harmonic Bergman space b^2 is the subspace of the Lebesgue space L^2 consisting of all complex-valued L^2-harmonic functions on D. One can check the relation $b^2 = L^2_a + \overline{L^2_a}$, where L^2_a denotes the holomorphic Bergman space consisting of all L^2-holomorphic functions on D. As is well known, the harmonic Bergman space b^2 is a closed subspace of L^2 and hence is a Hilbert space. We will write Q for the Hilbert space orthogonal projection from L^2 onto b^2. Each point evaluation is easily verified to be a bounded linear functional on b^2. Hence, for each $z \in D$, there exists a unique function R_z—called the harmonic Bergman kernel—in b^2 that has the following reproducing property:

$$u(z) = \langle u, R_z \rangle \tag{1}$$

for every $u \in b^2$. Here and elsewhere, the notation $\langle \cdot, \cdot \rangle$ denotes the usual inner product in L^2. Since $b^2 = L^2_a + \overline{L^2_a}$, there is a simple relation between the harmonic Bergman kernel R_z and the well-known (holomorphic) Bergman kernel K_z: $R_z = K_z + \overline{K_z} - 1$. Thus, the explicit formula of R_z is given by

$$R_z(w) = \frac{1}{(1 - w\bar{z})^2} + \frac{1}{(1 - \bar{w}z)^2} - 1 \quad (w \in D). \tag{2}$$

The formulas (1) and (2) lead us to the following integral representation of the projection Q:

$$Q\varphi(z) = \int_D \left(\frac{1}{(1 - z\bar{w})^2} + \frac{1}{(1 - \bar{z}w)^2} - 1 \right) \varphi(w)\, dA(w) \quad (z \in D) \tag{3}$$

for functions $\varphi \in L^2$. See [ABR, Chap. 8] for more information and related facts.

For $u \in L^2$, the *Toeplitz operator* T_u with *symbol* u is defined by

$$T_u f = Q(uf)$$

for functions $f \in b^2$. The operator T_u is densely defined and not bounded in general.

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