The Bergman Kernel on Monomial Polyhedra

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0. Introduction

In order to understand the Bergman kernel for a complex domain Ω in $\mathbb{C}^n$ at z close to the boundary $\partial\Omega$, we usually insert the biholomorphic image of a polydisc $\mathcal{D}$ centered at z in Ω to generate the upper bound for the Bergman kernel on Ω:

$$K_\Omega(z,z) \le K_{\mathcal{D}}(z,z) = \frac{1}{\mathrm{Vol}(\mathcal{D})}.$$

On the other hand, Catlin [3] showed by using a $\bar\partial$ estimate that, on a finite type pseudoconvex domain Ω in $\mathbb{C}^2$, there exists a polydisc $\mathcal{D}$ such that

$$K_\Omega(z,z) \ge c \cdot \frac{1}{\mathrm{Vol}(\mathcal{D})};$$

the same formula was later shown by McNeal [8] on convex domains in $\mathbb{C}^n$. A question arises: Are polydiscs enough to describe the Bergman kernel for smooth bounded domains?

For a general domain in $\mathbb{C}^n$, it is not always possible to find a polydisc D that models the domain. Consider $\Omega \subset \mathbb{C}^3$ defined by $|z_1|^{10} + |z_2|^{10} + |z_1z_2|^2 + |z_3|^2 < 1$, and let $z = (0,0,1-\varepsilon)$. It is easy to show that all polydiscs centered at z in Ω have maximal volume of approximately ε^4; thus, the upper bound of the Bergman kernel at z obtained by inserting polydiscs is roughly ε^{-4}. But consider a Reinhardt domain $\mathcal{R}$ centered at z bounded by $|z_1| < 1$, $|z_2| < 1$, $|z_3 - (1-\varepsilon)| < \varepsilon/2$, and $|z_1z_2| < \varepsilon/2$. The volume of $\mathcal{R}$ is roughly $\varepsilon^4(-\log\varepsilon + 1)$, which is much larger than ε^4 when $\varepsilon \ll 1$; therefore, the upper bound at z given by $\mathcal{R}$ is $1/\varepsilon^4(-\log\varepsilon + 1)$, much smaller than the ones given by any polydiscs.

The preceding example shows that polydiscs do not provide a good enough way of estimating upper bounds for the Bergman kernel. Instead of trying to fit a polydisc $\mathcal{D}$ about the point z into Ω, it seems better to try to fit the largest "monomial polyhedron" P about z into Ω, where a monomial polyhedron P associated with a finite subcollection $\mathcal{B}$ of index space $\mathcal{N}^n$, $\mathcal{N} = \mathbb{N} \cup \{0\}$, is defined as follows.

Definition 1.1. A domain P in $\mathbb{C}^n$ is a *monomial polyhedron* if there exists a subset $\mathcal{B} = \{\alpha_1, \dots, \alpha_m\}$ of $\mathcal{N}^n$ and, for each $\alpha \in \mathcal{B}$, there exists a unique $C_\alpha \in \mathbb{R}$ such that $P = P(\mathcal{B}) = \{z \in \mathbb{C}^n : |z^\alpha| < e^{C_\alpha}, \alpha \in \mathcal{B}\}$.

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