Injectivity and the Pre-Schwarzian Derivative

DENNIS STOWE

Many basic theorems about conformal mapping involve the pre-Schwarzian derivative f''/f'. This paper studies the inner radius of injectivity $\tau(D)$ of a simply connected domain D in the complex plane, other than the plane itself, with respect to that operator. In answer to questions posed by Gehring [9], we show that $\tau(D)$ never exceeds $1/2$ and that it equals $1/2$ for some domains other than disks and half-planes. We also show that every such domain is convex.

Let $\rho_D|dz|$ be the hyperbolic metric of D. When D is the unit disk, for example, $\rho_D(z)$ equals $2/(1-|z|^2)$, and when D is the right half-plane $\rho_D(x+iy)$ equals $1/x$. The inner radius of injectivity $\tau(D)$ is defined as the supremum of all numbers $c \geq 0$ such that every analytic function f in D satisfying the bound $|f''/f'| \leq c\rho_D$ is injective.

In the case of a disk or half-plane, τ is known to equal $1/2$. One part of the argument is due to Becker [4], who proves that $\tau \geq 1/2$ for the unit disk B. In fact, he proves a stronger result: An analytic function f in B is injective if $f'(0) \neq 0$ and

$$\left| z \cdot \frac{f''}{f'}(z) \right| \leq \frac{1}{1-|z|^2}, \quad z \in B.$$

A second ingredient is due to Becker and Pommerenke [5], who show that $\tau \leq 1/2$ for the right half-plane H. Citing an observation by Gehring, those authors conclude that equality holds in both instances. Indeed, the general formula

$$\frac{(f \circ h)''}{(f \circ h)'}(z) = \frac{h''}{h'}(z) + h'(z) \cdot \frac{f''}{f'}(h(z))$$

implies that τ is invariant under affine transformations from one domain onto another. Since any two points in H are contained in a disk that is in turn contained in H, it follows from the Schwarz lemma that $\tau(B) \leq \tau(H)$. Both quantities therefore equal $1/2$, and the conclusion extends to any disk or half-plane.

Gehring points out many parallels between $\tau(D)$ and the inner radius of injectivity $\sigma(D)$ with respect to the Schwarzian derivative $S(f) = (f''/f')' - (f''/f')^2/2$. The latter is defined as the supremum of all numbers $c \geq 0$ such that every analytic function f in D satisfying $|S(f)| \leq c\rho_D^2$ is injective. Both quantities are positive for quasidisks and zero otherwise; Martio and Sarvas [14] and Astala and Gehring [3] prove that result for τ, and Ahlfors [1] and Gehring [8] prove it

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