A Counterexample Related to Hartogs' Phenomenon (A Question by E. Chirka)

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We will denote by U (resp. $\bar{U}$) the open (resp. closed) unit disk in $\mathbb{C}$. Chirka [1] (see also [2]) recently proved the following remarkable result.

THEOREM (Chirka). *Let f be a continuous function on $\bar{U}$ with values in U, and let S be its graph ($S = \{(\zeta, f(\zeta)) \in \mathbb{C}^2, \zeta \in \bar{U}\}$). Then every holomorphic function defined on a connected neighborhood of the set $(\partial U \times U) \cup S$ in $\mathbb{C} \times U$ extends holomorphically to the polydisk U^2.*

It is shown by a simple example in [1] that the condition $|f| < 1$ on U (not only on ∂U) is essential.

If f is holomorphic, the result is of course classical. Here, answering a question by Chirka, we show that surprisingly (?) the theorem just stated does not extend to higher dimensions.

Our result is as follows.

PROPOSITION. *There exist continuous functions φ_1, φ_2 defined on $\bar{U}$ and satisfying $|\varphi_1|, |\varphi_2| < 1$, and there exists a domain ω in $\mathbb{C}^3$ such that:*

(i) *ω contains $\partial U \times U^2$ and ω contains the graph of (φ_1, φ_2) (i.e., $(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \in \omega$ for every $\zeta \in \bar{U}$); but*

(ii) *there exists a holomorphic function h on ω that does not extend holomorphically to U^3.*

REMARK. It may be worthwhile pointing out that, in the construction detailed next, the following is achieved: One can find an arbitrarily small neighborhood Z of $\partial U \times U^2$ and functions φ_1 and φ_2, as in the Proposition, such that the union of Z and of the graph of (φ_1, φ_2) has a basis of pseudoconvex neighborhoods.

An explicit example would still be desirable. The first and main step in the construction of the example is as follows. Find a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^3$, as well as the graph Γ of a smooth function on the unit disk, $\Gamma = \{(\zeta, \varphi_1(\zeta), \varphi_2(\zeta)) \in \mathbb{C}^3, \zeta \in \bar{U}\}$, such that the following statements hold.

(a) Ω contains $\partial U \times U^2$.

(b) $|\varphi_1|$ and $|\varphi_2| < 1$, and $|\partial\varphi_1/\partial\bar{\zeta}| + |\partial\varphi_2/\partial\bar{\zeta}| \neq 0$.

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