Invertibility Preserving Maps Preserve Idempotents

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Introduction and Statement of Main Results

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras. A linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is called *unital* if $\phi(1) = 1$ and is called *invertibility preserving* if $\phi(a)$ is invertible in $\mathcal{B}$ for every invertible element $a \in \mathcal{A}$. Similarly, ϕ preserves idempotents if $\phi(p)$ is an idempotent whenever $p \in \mathcal{A}$ is an idempotent; it is called a *Jordan homomorphism* if $\phi(a^2) = (\phi(a))^2$ for every $a \in \mathcal{A}$.

In [14, Sec. 9] Kaplansky asked: When must unital surjective linear invertibility preserving maps be Jordan homomorphisms? This problem was motivated by the famous Gleason–Kahane–Żelazko theorem [9; 13; 17], which states that every unital invertibility preserving linear map from a Banach algebra to a semisimple commutative Banach algebra is multiplicative, as well as by results of Dieudonné [8] and Marcus and Purves [15] stating that every unital invertibility or singularity preserving linear map on a matrix algebra is either multiplicative or antimultiplicative. The case of a nonunital invertibility preserving mapping can be reduced to the unital case by considering θ defined by $\theta(a) = \phi(1)^{-1}\phi(a)$.

The answer to Kaplansky's question is not always affirmative. Some historical remarks on this problem can be found in [1, pp. 27–31], where the first noncommutative extensions of the Gleason–Kahane–Żelazko theorem were mentioned. Having in mind all known results and counterexamples, it is tempting to conjecture that the answer to Kaplansky's question is affirmative if $\mathcal{A}$ and $\mathcal{B}$ are semisimple Banach algebras [3; 10; 16].

Let X be a Banach space, and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. By $\mathcal{F}(X)$ we denote the ideal of all finite rank operators. For every $x \in X$ and every bounded linear functional f on X, we denote by $x \otimes f$ a rank-1 operator defined by $(x \otimes f)y = f(y)x$. Following Chernoff [7], we call a subalgebra $\mathcal{A} \subset \mathcal{B}(X)$ a *unital standard operator algebra* on X if it is closed and contains I and $\mathcal{F}(X)$. We will prove that the problem of characterizing linear invertibility preserving mappings can be reduced to the problem of characterizing linear maps preserving idempotents if the codomain is a unital standard operator algebra.

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