Analytic Varieties with Boundaries in Totally Real Tori

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1. Introduction

Let ∂D be the unit circle in $\mathbf{C}$ and let $\pi_2 \colon \mathbf{C}^2 \to \mathbf{C}$ be the projection $\pi_2(z, w) = w$. Let T_1 and T_2 be disjoint maximal real smooth tori in $\partial D \times \mathbf{C}$ such that, for each $\xi \in \partial D$ and $j = 1, 2$, the fiber

$$T_{j,\xi} := \pi_2(T_j \cap (\{\xi\} \times \mathbf{C}))$$

of T_j over ξ is a smooth Jordan curve in $\mathbf{C}$. Also, let V be a two-sheeted analytic variety over D with boundary in $T_1 \cup T_2$; that is, there exist p and q holomorphic functions on D, continuous up to the boundary ∂D, such that

$$V = \{ (z, w) \in \bar{D} \times \mathbf{C};\ w^2 - p(z)w + q(z) = 0 \}$$

and such that, for every boundary point $\xi \in \partial D$, each curve $T_{1,\xi}$ and $T_{2,\xi}$ contains exactly one root of the equation $w^2 - p(\xi)w + q(\xi) = 0$.

In this paper we consider the question of when it is possible to perturb variety V along $T_1 \cup T_2$. More precisely, we are interested in geometric conditions on $T_1 \cup T_2$ and V such that it is possible to parameterize all nearby two-sheeted varieties over D with boundaries in $T_1 \cup T_2$. The method we apply is the method of partial indices, which has been successfully used in problems of perturbing analytic discs along maximal real boundaries by several authors [4; 6; 7; 8; 9]. The geometric conditions we obtain are expressed in terms of the winding numbers of the normals to the fibers $T_{j,\xi}$ ($j = 1, 2$) along the roots of the equation $w^2 - p(\xi)w + q(\xi) = 0$ ($\xi \in \partial D$). A typical result is the following.

Theorem. *Let V be an irreducible two-sheeted analytic variety over D with boundary in the disjoint union $T_1 \cup T_2$ of two maximal real tori fibered over ∂D. Let α_1 and α_2 be the complex functions on ∂D representing the boundary roots of the variety V such that, for every $\xi \in \partial D$, we have $\alpha_j(\xi) \in T_{j,\xi}$ ($j = 1, 2$); let $\Delta = \alpha_1 - \alpha_2$. Also, let $\nu_1(\xi)$ and $\nu_2(\xi)$ be normals to the fibers $T_{1,\xi}$ and $T_{2,\xi}$ at the points $\alpha_1(\xi)$ and $\alpha_2(\xi)$, respectively. If $W(\nu_1) + W(\nu_2) \geq -1$, then the family of two-sheeted analytic varieties over D with boundaries in $T_1 \cup T_2$ that are close to V is a C^1 submanifold of the space of two-sheeted analytic varieties over D of dimension* $2(W(\nu_1) + W(\nu_2) + W(\Delta)) + 2$.

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