New Examples of Homogeneous Einstein Metrics

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1. Introduction

A Riemannian metric is said to be *Einstein* if the Ricci curvature is a constant multiple of the metric. Given a manifold M, one can ask whether M carries an Einstein metric and, if so, how many. This fundamental question in Riemannian geometry is for the most part unsolved (cf. [Bes]). As a global PDE or a variational problem, the question is intractible. It becomes more manageable in the homogeneous setting, and so many of the known examples of compact simply connected Einstein manifolds are homogeneous. In this paper we give a technique for finding and classifying all homogeneous metrics on any given homogeneous space, including those that are not diagonal with respect to the isotropy representation. We also examine some compact simply connected homogeneous spaces G/H, where G is simple and H is closed and connected. On each space we describe all G-invariant Einstein metrics. For such spaces, the normal homogeneous Einstein metrics were classified by Wang and Ziller [WZ1]. Among the metrics we shall find, there is only one normal metric: the metric on $S^7 \times S^7$ induced by the Killing form. In fact, apart from $S^7 \times S^7$, none of our examples of homogeneous Einstein metrics is even naturally reductive.

Each of our examples has G-invariant metrics that are not diagonal with respect to the isotropy representation of H. Few examples of this type have been previously examined. Some nondiagonal examples arise as fibrations with Riemannian submersion metrics, where the base and fibre are Einstein—for example, if the base and fibre are irreducible symmetric spaces. Using this method, we can expect a product Einstein metric on each of the examples to follow. Jensen does this to find a homogeneous Einstein metric on Stiefel manifolds $V_k\mathbb{R}^n$. He restricts to a two-parameter family of diagonal $\mathrm{SO}(n)$-invariant metrics on $V_k\mathbb{R}^n$ [J2]. Using very different methods, Sagle also considers Stiefel manifolds, showing that $V_k\mathbb{R}^n$ carries at least one Einstein metric [S]. Sagle first discovered the $\mathrm{SO}(n)$-invariant Einstein metric on $V_2\mathbb{R}^n$. Neither Sagle nor Jensen observes that the homogeneous Einstein metric on $V_2\mathbb{R}^n$ is unique. More recently, Arvanitoyeorgos looks at a special family of $\mathrm{SO}(n)$-invariant metrics on $V_k\mathbb{R}^n$ [A]. None of these methods exhausts all possible homogeneous Einstein metrics.

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