New Cases of Almost Periodic Factorization of Triangular Matrix Functions

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1. Introduction

A function f is said to be an *almost periodic polynomial* if it can be expressed in the form

$$f(x) = \sum_{j=1}^{m} c_j e^{i\lambda_j x} \quad \text{with } c_j \in \mathbb{C} \text{ and } \lambda_j \in \mathbb{R}. \tag{1.1}$$

The set of all almost periodic polynomials forms an algebra AP_P. The closure of AP_P under the uniform norm $\|f\| = \sup_{x\in\mathbb{R}}|f(x)|$ gives the algebra AP of almost periodic functions. In other words, AP is the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by all functions $e_\lambda(x) = e^{i\lambda x}$, $\lambda \in \mathbb{R}$.

The *mean value* of an almost periodic function is defined as

$$\mathbf{M}(f) = \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} f(x)\,dx,$$

and the *Fourier coefficient* $\mathbf{M}_\lambda(f) := \mathbf{M}(e_{-\lambda} f)$. (These definitions are standard; see [4] and [14].) Of course, $\mathbf{M}(f) = \mathbf{M}_0(f)$. For $f \in AP_P$ written in the form (1.1), $\mathbf{M}_{\lambda_j}(f) = c_j$.

The *Fourier spectrum* of f, denoted $\Omega(f)$, is defined as $\{\lambda \in \mathbb{R} : \mathbf{M}_\lambda(f) \neq 0\}$.

We use AP^+ (resp. AP^-) to denote the subalgebra consisting of all $f \in AP$ such that $\Omega(f) \subset [0,\infty)$ (resp. $(-\infty, 0]$). A matrix function is said to be in AP or in $AP^\pm$ if all of its entries are. We say that an $n \times n$ matrix AP function G is *AP-factorable* if it can be represented as a product

$$G(x) = G^+(x)\Lambda(x)G^-(x), \tag{1.2}$$

where $(G^+)^{\pm 1} \in AP^+$, $(G^-)^{\pm 1} \in AP^-$, and $\Lambda = \operatorname{diag}[e_{\lambda_1}, \dots, e_{\lambda_n}]$, $\lambda_j \in \mathbb{R}$. Factorization (1.2) was introduced in [10]. It was also observed there that, if G is periodic with a period T, then a simple change of variable $t = e^{ixT/2\pi}$ reduces

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