The Hardy Class of Kœnigs Maps

PIETRO POGGI-CORRADINI

1. Introduction

An analytic function ϕ on $\mathbb{D}$ such that $\phi(\mathbb{D}) \subset \mathbb{D}$ induces a linear (composition) operator $C_\phi(f) = f \circ \phi$ on the functions f defined on $\mathbb{D}$. The operator C_ϕ is bounded on the Hilbert space $H^2(\mathbb{D})$ of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty \tag{1.1}$$

In this article we restrict our attention to the case when ϕ fixes a point in the disk and has nonzero derivative there. Without loss of generality we can conjugate such a fixed point to the origin with a Möbius transformation; to avoid trivial situations, we assume that ϕ is not an automorphism of $\mathbb{D}$.

DEFINITION 1.1. We consider the family $\mathcal{A}$ of functions ϕ that are analytic on $\mathbb{D}$, with $\phi(\mathbb{D}) \subset \mathbb{D}$, $\phi(0) = 0$, and $0 < |\phi'(0)| < 1$.

For each function $\phi \in \mathcal{A}$, Kœnigs's theorem (see Proposition on p. 91 of [Sh2]) provides a function σ analytic on $\mathbb{D}$, with $\sigma(0) = 0$ and $\sigma'(0) \neq 0$, that solves Schröder's equation:

$$\sigma \circ \phi(z) = \lambda \sigma(z) \qquad \forall z \in \mathbb{D} \tag{1.2}$$

with $\lambda = \phi'(0)$.

Near the origin, $\phi \sim \lambda z$ and σ conjugates ϕ to λz (since σ is one-to-one in a neighborhood of zero). However, σ is defined in the whole unit disk and, by (1.2), it intertwines the action of ϕ on $\mathbb{D}$ with multiplication by λ on $\mathbb{C}$. Therefore, the growth of σ near $\partial\mathbb{D}$ should encapsulate the repelling properties of ϕ near $\partial\mathbb{D}$.

We first determine how we are going to measure the growth of σ. Recall that, for each $p > 0$, one defines the Hardy space $H^p(\mathbb{D})$ of analytic functions on $\mathbb{D}$ satisfying a growth condition as in (1.1) by replacing $|f(re^{i\theta})|^2$ with $|f(re^{i\theta})|^p$, and the Nevanlinna class $\mathcal{N}(\mathbb{D})$ by replacing $|f(re^{i\theta})|^2$ with $\log^+|f(re^{i\theta})|$. Then $H^{p_1}(\mathbb{D}) \subset H^{p_2}(\mathbb{D})$ for $p_1 > p_2$, and $\bigcup_{p>0} H^p(\mathbb{D})$ is strictly contained in $\mathcal{N}(\mathbb{D})$. For every analytic function f on $\mathbb{D}$ we set $h(f) = \sup\{p > 0 : f \in H^p(\mathbb{D})\} \in [0, \infty]$ and call it the *Hardy number* of f. Clearly, $f \in H^p(\mathbb{D})$ when $0 < p < h(f)$ and $f \notin H^p(\mathbb{D})$ when $h(f) < p < \infty$; however, at issue is determining

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