Projectivity and Extensions of Hilbert Modules over $\mathbb{A}(\mathbb{D}^N)$

JON F. CARLSON & DOUGLAS N. CLARK

1. Introduction

Let $\mathbb{A}(\mathbb{D}^N)$ denote the algebra of functions analytic in the polydisk $\mathbb{D}^N$ that extend continuously to its closure, and let $\mathcal{H}$ be the category of all Hilbert $\mathbb{A}(\mathbb{D}^N)$-modules. Briefly, an *object* in $\mathcal{H}$ is a Hilbert space H, together with a continuous bilinear multiplication $\mathbb{A}(\mathbb{D}^N) \times H \to H$. Douglas and Paulsen [5] were the first to approach the category $\mathcal{H}$ as a natural setting for certain questions of operator theory. For example, considering the *operator variables,* the operators of multiplication by $z_1, \dots, z_N$ in H, the transition from 1 to N commuting operators can be viewed as a natural one. In the present note, we deal with several questions about Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$ that have been previously answered for $N = 1$.

One of the problems left open by Douglas and Paulsen [5] was that of finding projective objects in the cateogry $\mathcal{H} = \mathcal{H}(\mathbb{A})$ for any function algebra $\mathbb{A}$. An answer in the case that $\mathbb{A} = \mathbb{A}(\mathbb{D})$ was given in [2], where it was proved that unitary Hilbert modules (Hilbert modules where the operator variable is unitary) are always projective. One purpose of the present note is to obtain an extension of the result to modules over $\mathbb{A}(\mathbb{D}^N)$ (see Theorem 3.1).

A second question of importance is whether the category $\mathcal{H}$ has *enough* projectives, in the sense that every object in $\mathcal{H}$ is a quotient of some projective module. Another result of [2] shows that the category $\mathcal{C}$ of *cramped* Hilbert modules—that is, Hilbert modules similar to contractive ones—has enough projectives. Specifically, it was proved that isometric Hilbert modules are projective in $\mathcal{C}$ and that the Sz.-Nagy–Foiaş model for a completely nonunitary contractive Hilbert module over $\mathbb{A}(\mathbb{D})$ gives a projective resolution. On the other hand, Pisier's recent example ([7], see also [4]) demonstrates that the isometric Hilbert modules are not in general projective in $\mathcal{H}(\mathbb{A}(\mathbb{D}))$. In particular, the vector-valued Hardy space $\mathbb{H}^2(H)$ is not projective, though it remains unknown whether non–vector-valued $\mathbb{H}^2$ is projective in $\mathcal{H}$. The examples make it seem doubtful that $\mathcal{H}(\mathbb{A}(\mathbb{D}))$ has enough projectives, though there is not yet any proof.

For $\mathbb{A}(\mathbb{D}^N)$-modules, the situation is worse. In the cramped category $\mathcal{C}(\mathbb{A}(\mathbb{D}^N))$, the best that can be proved is that, if one of the operator variables is an isometry and all of the others are unitary, then the object is projective (Theorem

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