Nonpositively Curved, Piecewise Euclidean Structures on Hyperbolic Manifolds

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A well-known question is whether any Riemannian manifold M of nonpositive sectional curvature admits a piecewise Euclidean metric that is nonpositively curved. Here "nonpositively curved" is in the sense of Aleksandrov and Gromov—that is, it is defined by comparing small triangles in the space with triangles in the Euclidean plane via the "CAT(0)-inequality". (See [BH] or [G] for the precise definition.) Our purpose in this paper is to describe a simple construction that gives an affirmative answer to the question in the case of constant sectional curvature.

The most naive approach to this problem does not work, at least not obviously. Namely, given a hyperbolic manifold, first find a triangulation of it by hyperbolic simplices. Next, replace each hyperbolic simplex by a Euclidean simplex with the same edge lengths. Finally, try to prove that the resulting metric is nonpositively curved. This approach works in dimension 2. However, in higher dimensions there are at least two problems with it: (1) there are hyperbolic simplices that do not have the same set of edge lengths as a Euclidean simplex (consider a hyperbolic tetrahedron with one face a big triangle and fourth vertex almost on the plane of the triangle); and (2) even when the replacement process can be carried out, the dihedral angles in the Euclidean simplex can be smaller than the corresponding dihedral angles in the hyperbolic simplex, so the curvature can become positive. We shall take a different tack.

We use the quadratic form model of hyperbolic n-space: $\mathbb{R}^{n,1}$ denotes an $(n+1)$-dimensional vector space with coordinates $(x_1, \dots, x_{n+1})$ equipped with the indefinite symmetric bilinear form $\langle\ ,\ \rangle$ defined by $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}$, and the associated quadratic form $q(x) = \langle x, x\rangle$. Hyperbolic space $\mathbb{H}^n$ is identified with the sheet of the hyperboloid $q(x) = -1$ defined by $x_{n+1} > 0$. If T is a k-dimensional linear subspace of $\mathbb{R}^{n,1}$, then there are three possibilities for the restriction of the bilinear form to T: either it is positive definite, positive semidefinite, or indefinite of signature $(k-1, 1)$. One says that T is, respectively, *spacelike, lightlike*, or *timelike*. If F is a k-dimensional convex subset of $\mathbb{R}^{n,1}$, then let T_F denote the k-dimensional linear subspace that is parallel to the affine span of F. We say that F is spacelike, lightlike, or timelike as T_F is.

Let V be a discrete subset of $\mathbb{H}^n$. The *Dirichlet region* D_v for V at a point v in V consists of the points in $\mathbb{H}^n$ which are at least as close to v as to $V - \{v\}$.

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