Harmonic Cohomology Classes of Almost Cosymplectic Manifolds

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1. Introduction

The concept of Poisson structure plays an important role in mathematics and physics. Apparently, Poisson structures in local coordinates were first considered in 1875 in the work of Lie [20]; from the mathematical viewpoint, such a theory has been developed since the early 1970s by Lichnerowicz [19], Weinstein [26], and others. A Poisson manifold is a smooth manifold M endowed with a Poisson bracket, that is, a Lie bracket $\{\ ,\ \}$ on the algebra of smooth functions on M satisfying Leibniz's rule. The existence of a Poisson bracket on M is equivalent to the existence of a skew-symmetric contravariant 2-tensor G on M satisfying $[G, G] = 0$, where $[\ ,\ ]$ denotes the Schouten–Nijenhuis bracket [1].

For a Poisson manifold M, Koszul [15] introduced a differential operator Δ: $\Lambda^k(M) \to \Lambda^{k-1}(M)$, defined by $\Delta = [i(G), d]$, where $i(G)$ denotes the contraction by G and d is the exterior derivative of M. We call it the Koszul differential and shall write δ instead of Δ. Since $\delta^2 = 0$ [4; 15], it defines the so-called canonical homology of M. Moreover, as in the Riemannian case, a Poisson Laplacian $\Delta = d\delta + \delta d$, which is identically zero, can be defined [15]. A k-form α is called harmonic (with respect to the Poisson structure) if $d\alpha = \delta\alpha = 0$. In [4], Brylinski proposed the following.

PROBLEM. Give conditions on a compact Poisson manifold M ensuring that any de Rham cohomology class has a harmonic (with respect to the Poisson structure) representative α, that is, $d\alpha = \delta\alpha = 0$.

In the particular case of symplectic manifolds, this problem has already been solved [4; 8; 21]. More precisely, Brylinski [4] proved that for compact Kähler manifolds this problem has an affirmative solution. However, we exhibit in [8] an example of a compact symplectic manifold M^4 and a de Rham cohomology class $\mathfrak{a}$ on M^4 such that $\mathfrak{a}$ does not admit harmonic representatives. Independently, Mathieu [21] proved that a compact symplectic manifold has the conditions of Brylinski's problem if and only if it satisfies the hard Lefschetz theorem.

Almost cosymplectic manifolds are another important class of Poisson manifolds. Remember that an almost cosymplectic manifold is a $(2n + 1)$-dimensional

Received April 1, 1996.
This work has been partially supported through grants DGICYT, Project PB94-0633-C02-02.
Michigan Math. J. 44 (1997).