The Moduli of Holomorphic Functions in Lipschitz Spaces

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1. Introduction and Results

Given $0 < \alpha < 1$, let Λ^α denote the classical *Lipschitz space* of the real line $\mathbb{R}$, that is, the set of all complex-valued functions $f \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$|f(t_1) - f(t_2)| \le \text{const}\,|t_1 - t_2|^\alpha, \quad t_1, t_2 \in \mathbb{R}$$

(the constant on the right may depend only on f). Further, let Λ^α_A stand for the corresponding *analytic subspace* consisting of those functions in Λ^α whose harmonic extensions (Poisson integrals) are holomorphic on

$$\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

In other words, elements of Λ^α_A are just H^∞ functions with boundary values in Λ^α (as usual, H^∞ denotes the algebra of bounded holomorphic functions on $\mathbb{C}_+$).

The problem we treat here is to characterize the absolute values of Λ^α_A functions. More precisely, given a nonnegative function φ on $\mathbb{R}$, we are concerned with explicit conditions under which φ agrees with (the boundary values of) the modulus $|f|$ of some function $f \in \Lambda^\alpha_A$.

The two immediate necessary conditions are

$$\varphi \in \Lambda^\alpha \tag{1.1}$$

and, if we exclude the trivial function $\varphi \equiv 0$ from consideration,

$$\int_{-\infty}^{\infty} \frac{\log \varphi(t)}{1+t^2}\, dt > -\infty. \tag{1.2}$$

In connection with (1.2), see [G, Chap. II, Sec. 4].

Once (1.2) holds, we form the *outer function* $\mathcal{O}_\varphi$ with modulus φ by setting

$$\mathcal{O}_\varphi(z) \stackrel{\text{def}}{=} \exp\left\{\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)\log \varphi(t)\, dt\right\}, \quad z \in \mathbb{C}_+,$$

and note that the above problem is equivalent to ascertaining when

$$\mathcal{O}_\varphi \in \Lambda^\alpha_A. \tag{1.3}$$

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