Capacity Distortion by Inner Functions in the Unit Ball of $\mathbf{C}^n$

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1. Introduction

An inner function is a bounded holomorphic function from the unit ball $\mathbf{B}_n$ of $\mathbf{C}^n$ into the unit disk Δ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If E is a nonempty Borel subset of $\partial\Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $\mathbf{S}_n$ of $\mathbf{C}^n$:

$$f^{-1}(E) = \{\xi \in \mathbf{S}_n : \lim_{r\to 1} f(r\xi) \text{ exists and belongs to } E\}.$$

There is a classical lemma of Löwner (see e.g. [R, p. 405; T, p. 322]), about the distortion of boundary sets under inner functions.

LÖWNER'S LEMMA. *An inner function f, with $f(0) = 0$, is a measure-preserving transformation when viewed as a mapping from $\mathbf{S}_n$ to $\partial\Delta$. That is, if E is a Borel subset of $\partial\Delta$ then $|f^{-1}(E)| = |E|$, where in each case $|\cdot|$ denotes the corresponding normalized Lebesgue measure.*

Here we extend this result to fractional dimensions as follows.

THEOREM 1. *Let f be inner in the unit ball of $\mathbf{C}^n$ $(n \geq 1)$, set $f(0) = 0$, and let E be a Borel subset of $\partial\Delta$. Then:*

(i) *if $0 < \alpha < 2$ (and also $\alpha = 0$ if $n = 1$), then*

$$\operatorname{cap}_{2n-2+\alpha}(f^{-1}(E)) \geq C(n,\alpha)\operatorname{cap}_\alpha(E); \tag{1.1}$$

(ii) *if $\alpha = 0$ and $n > 1$, then*

$$\frac{1}{\operatorname{cap}_{2n-2}(f^{-1}(E))} \leq C(n)\left(1 + \log\frac{1}{\operatorname{cap}_0(E)}\right). \tag{1.2}$$

Here $\operatorname{cap}_\alpha$ and cap_0 denote (respectively) α-dimensional Riesz capacity and logarithmic capacity with respect to the distance in $\mathbf{S}_n$ given by

$$d(a,b) = |1 - \langle a,b\rangle|^{1/2},$$

where

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