Transverse Heegaard Splittings

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1. Background and the Theorem

In [RS] we use Cerf theory to compare irreducible Heegaard splittings of the same irreducible non-Haken orientable 3-manifold. A critical part of the argument is the observation that any two Heegaard surfaces may be isotoped so that they intersect in a nonempty collection of simple closed curves, each of which is essential in both surfaces. Here we describe an analog to this theorem that applies to the Haken case. An eventual goal, not yet realized here, is a bound for the number of stabilizations needed to make two distinct Heegaard splittings equivalent. Such a bound is found, for the non-Haken case, in [RS].

All manifolds are assumed to be compact and orientable. It is a simple and standard exercise to show that, if S and T are closed incompressible surfaces in an irreducible 3-manifold M, then they can be isotoped so their intersection (if any) is a collection of simple closed curves, each of which is essential in both S and T. A Heegaard surface in M is as unlike an incompressible surface as possible. It is a surface that is not only compressible, but a surface that can be compressed entirely away—on both sides. Yet it is shown in [RS] that, if M is closed and non-Haken, then a pair of Heegaard surfaces behave something like a pair of incompressible surfaces: Any pair of Heegaard surfaces P and Q can be isotoped so that they intersect in a nonempty collection of simple closed curves, each of which is essential in both P and Q. The content here is in the word "nonempty", since it is obvious that P and Q can be made disjoint: Choose disjoint spines of handlebodies bounded by P and Q, then isotope P and Q near the respective spines. The purpose here is to extend this result, in a somewhat different form, to the case in which M may be Haken.

A *compression body* H is constructed by adding 2-handles to a (surface) $\times I$ along a collection of disjoint simple closed curves on (surface) $\times \{0\}$, and capping off any resulting 2-sphere boundary components with 3-balls. The component (surface) $\times \{1\}$ of ∂H is denoted $\partial_+ H$, and the surface $\partial H - \partial_+ H$ is denoted $\partial_- H$. If $\partial_- H = \emptyset$ then H is a *handlebody*. If $H = \partial_+ H \times I$ then H is called a *trivial* compression body. Define the *index* $I(H)$ of H to be $\chi(\partial_- H) - \chi(\partial_+ H)$.

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