Generalized Roundness and Negative Type

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1. Introduction

In this paper we exhibit the equivalence of Enflo's nonlinear notion of generalized roundness and the classical embedding notion of negative type. This enables us to develop a rudimentary theory of generalized roundness and to give applications to the L_p-spaces. In particular, we show that for $p > 2$ and $n \geq 3$, the n-dimensional l_p spaces fail to have generalized roundness q for all $q > 0$.

The notions of roundness and generalized roundness were introduced by Enflo in [E1], [E2], and [E3] to study the uniform structure of metric spaces. We begin by recalling some material from these papers. However, we make some slight alterations to Enflo's original definitions to allow easier exposition later.

1.1. Definition. (a) We say that a metric space (X, d) has *roundness* q, written $q \in \mathrm{r}(X, d)$, if whenever a_1, a_2, b_1, b_2 are in X we have

$$d(a_1, a_2)^q + d(b_1, b_2)^q \leq \sum_{1 \leq i, j \leq 2} d(a_i, b_j)^q. \tag{1}$$

(b) A pair $(a_1, \dots, a_n), (b_1, \dots, b_n)$ of n-tuples in a metric space is called a *double-n-simplex*. Such a double-n-simplex will be denoted $[a_i; b_i]_{i=1}^n$. We call a pair of points (a_i, a_j) or (b_i, b_j) an *edge*, and a pair of points (a_i, b_j) a *connecting line*.

We say that a metric space (X, d) has *generalized roundness* q, written $q \in \mathrm{gr}(X, d)$, if for every $n \geq 2$ and every double-n-simplex $[a_i; b_i]_{i=1}^n$ in X we have

$$\sum_{1 \leq i < j \leq n} \big(d(a_i, a_j)^q + d(b_i, b_j)^q\big) \leq \sum_{1 \leq i, j \leq n} d(a_i, b_j)^q. \tag{2}$$

When (2) holds for a specific double-n-simplex $[a_i; b_i]_{i=1}^n$ in X we will write $q \in \mathrm{gr}[a_i; b_i]_{i=1}^n$.

1.2. Remark. In [E1] and [E2] Enflo defined the roundness of a metric space (X, d) to be $\sup\{\, q \mid q \in \mathrm{r}(X, d)\,\}$ and the generalized roundness to be $\sup\{\, q \mid q \in \mathrm{gr}(X, d)\,\}$. It is easy to check that $\{\, q \mid q \in \mathrm{r}(X, d)\,\}$ and $\{\, q \mid q \in \mathrm{gr}(X, d)\,\}$ are

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