An Inverse Function Theorem for Fréchet Spaces Admitting Generalized Smoothing Operators

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Introduction

An inverse function theorem of the Nash–Moser type is proved for Fréchet spaces admitting generalized smoothing operators; the proof is based on Newton's method. In particular, for Köthe sequence spaces, property (Ω) in the standard form and the topological condition (DN) in the sense of Vogt are shown to be sufficient for the Nash–Moser theorem to hold under classical assumptions on the mappings.

In the literature, inverse function theorems of so-called Nash–Moser type with "loss of derivatives" are proved for Fréchet spaces that admit smoothing operators as introduced by Nash [8]; a possible proof relies on Newton's method as suggested by Moser in [7] (see e.g. [2; 3; 5; 11; 12; 13] or [1; 6] for generalized results). For instance, Lojasiewicz and Zehnder [5] prove such a theorem showing that Newton's method still converges if the classical "tame" assumptions on the mappings (cf. [2]) are replaced by "linear-tame estimates with $1 \le \lambda < 2$" while the theorem fails if $\lambda = 2$ (cf. [5]). This paper contains a generalization of [5]; the aim is to find out under which more general conditions on the Fréchet space Newton's method converges. The hypothesis of smoothing operators is replaced by the weaker assumption of the existence of generalized smoothing operators, and the (linear-) tame estimates supposed in [5] are replaced by more general estimates. It is then considered as a property of the Fréchet space under which assumptions on the mappings the inverse function theorem holds. This property of the Fréchet space is quantitatively measured by means of the existence of suitable generalized smoothing operators.

The first section contains preliminaries. Section 2 treats the standardized case, "loss of derivatives = 1"; for this situation, a generalization of the result in [5] is proved. It is carefully checked which property of the Fréchet space is needed to compensate this loss of derivatives in order to make Newton's method converge. In [5], the existence of classical smoothing operators and hence property (DN) in standard form are assumed; here only the

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