On Removable Singularities for the Analytic Zygmund Class

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1. Introduction and Statement of Results

A complex-valued function f defined on the complex plane $\mathbb{C}$ belongs to the Zygmund class ($f \in \Lambda_*$), or quasismooth class, if it is bounded and there exists a positive constant C such that

$$|f(z+h)+f(z-h)-2f(z)| \leq C|h| \tag{1.1}$$

for all $z, h \in \mathbb{C}$.

The boundedness of f and (1.1) imply the continuity of f. We define the Zygmund norm as $\|f\|_* = \|f\|_\infty + \|f\|_{\Lambda_*}$, where $\|f\|_{\Lambda_*}$ denotes the smallest constant C for which (1.1) holds.

We shall call a compact subset K in $\mathbb{C}$ a removable set for the analytic functions of the Zygmund class (resp. Lipschitz class) provided that every function $f \in \Lambda_*$ (resp. $f \in \text{Lip}_\alpha$) that is analytic on $\mathbb{C}\setminus K$ has an analytic extension to the entire plane.

We recall the definition of Hausdorff measure. A measure function is an increasing continuous function $h(t)$, $t \geq 0$, such that $h(0) = 0$. Let E be a bounded set, and for $0 < \delta \leq \infty$ write

$$\Lambda_h^\delta(E) = \inf\left\{ \sum_{j=1}^{\infty} h(\text{diam}(U_j)) : E \subset \bigcup_{j=1}^{\infty} U_j,\ \text{diam}(U_j) \leq \delta \right\}.$$

Since $\Lambda_h^\delta(E)$ is a decreasing function of δ, the limit

$$\Lambda_h(E) = \lim_{\delta \to 0} \Lambda_h^\delta(E) = \sup_{\delta > 0} \Lambda_h^\delta(E)$$

exists; it is called the Hausdorff measure of E with respect to h. For instance, if $h(t) = t^\alpha$ for some $\alpha > 0$, then we will write Λ_α instead of Λ_h. We will denote by m the planar measure Λ_2. If $\delta = \infty$, $\Lambda_h^\infty = M_h$ is called the Hausdorff content with respect to h. From the definitions it follows that $\Lambda_h(E) = 0$ if and only if $M_h(E) = 0$. See [2] for more information.

Dolzenko [1] proved that K is removable for the analytic functions of Lip_α ($0 < \alpha < 1$) if and only if $\Lambda_{1+\alpha}(K) = 0$. This result is also true for the extreme case $\alpha = 1$, as was proved by Uy [11]. The limit case $\alpha = 0$ corresponds

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