An Isoperimetric-Type Inequality for Integrals of Green's Functions

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1. Introduction

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, with finite volume. Let D^* be the ball centered at the origin and of the same volume as D. Denote by $G_D(w,z)$ and $G_{D^*}(w,z)$ the Green's functions for D and D^*, respectively. The following isoperimetric inequality is now a classical result (see [1, p. 61]):

$$\sup_{w \in D} \int_D \varphi(G_D(w,z))\,dz \leq \int_{D^*} \varphi(G_{D^*}(0,z))\,dz \tag{1.1}$$

for all nonnegative nondecreasing functions φ defined on $[0,\infty)$.

However, for a large class of domains D and functions φ, what determines the finiteness of the quantity on the left-hand side of (1.1) is not the volume of the domain but rather its inner radius. This is true in particular for any simply connected domain in the plane. More precisely, if D is a simply connected domain in the complex plane we let R_D be the radius of the largest disc contained in D (if such a disc exists) and the limit superior of the radii of all discs contained in D otherwise. We call R_D the *inner radius* of D. Assume φ satisfies

$$\int_0^\infty r\varphi(\log(\coth(r))\,dr < \infty,$$

(e.g., $\varphi(x) = x^p$, $0 < p < \infty$), then by Bañuelos and Carroll [2] we have

$$\sup_{w \in D} \int_D \varphi(G_D(w,z))\,dz < \infty \tag{1.2}$$

if and only if $R_D < \infty$. It is then natural to inquire about the following extremal problem: Amongst all simply connected planar domains D with $R_D = 1$, find those that maximize the left-hand side of (1.2). This problem, which is wide open even for $\varphi(x) = x$, is closely related to an extremal problem for the lowest Dirichlet eigenvalue of D and to the well-known problem in function theory concerning the schlicht Bloch–Landau constant. We refer the reader to [2] for more on this connection. When we restrict ourselves to convex domains, it has been proved by Sperb [9, p. 87] that

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