The Polar Dual of a Convex Polyhedral Set in Hyperbolic Space

Ruth Charney & Michael Davis

Introduction

Let X be a convex polyhedral set in a space of constant curvature. Its *polar dual* $P(X)$ is the set of outward-pointing unit normal vectors to the supporting hyperplanes of X. In the spherical and Euclidean cases, $P(X)$ is a subset of the unit sphere. In the hyperbolic case, $P(X)$ is a subset of the unit pseudosphere in Minkowski space (sometimes called the "de Sitter sphere"). In all three cases $P(X)$ naturally has the structure of a piecewise spherical cell complex. The spherical cells of $P(X)$ correspond bijectively to the faces of X.

A piecewise spherical cell complex, with its intrinsic metric, is *large* if there is a unique geodesic between any two points of distance less than π. Equivalently, it is large if it satisfies Gromov's CAT(1)-inequality [G].

Piecewise spherical cell complexes play an important role in the study of certain singular metric spaces: the link (or "space of directions") of a point in such a singular space often has a piecewise spherical structure. The largeness condition is closely related to the notion of nonpositive curvature in the sense of Alexandrov and Gromov. For example, a polyhedron of piecewise constant curvature has curvature bounded from above if and only if the link of each point is large (cf. [G; B]). A similar result holds for the induced singular metric on the branched cover of a Riemannian manifold [CD1].

In 1986 Rivin [R1] proved that the polar dual of a convex polytope in hyperbolic 3-space is large. (This was published as the paper [HR] of Hodgson and Rivin.) The proof is a simple geometric argument; the main result of [HR] is in the converse direction. The referee has pointed out that some related results, in the smooth category, are proved in [S].

In 1988 Moussong [M] considered a related situation: he gave a simple, necessary and sufficient condition for a piecewise spherical, simplicial complex with all edge lengths $\geq \pi/2$ to be large. The results of both Rivin and Moussong are generalizations of Andreev's theorem [A].

In this paper we use Moussong's ideas to extend Rivin's argument to any hyperbolic convex polyhedral set (not necessarily compact) of any dimension. The main result is the following.

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