# On Virtually Projective Groups

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## 1. Introduction

Denote the maximal pro-2 Galois group of a field $K$ by $G_K(2)$. Fields $K$ for which $G_{K(\sqrt{-1})}(2)$ is a free pro-2 group were studied by various authors (see e.g. [ELP; Er; W1; W2; W3; E1]). The approach in these works is, to a large extent, arithmetical—leaning heavily on (among other things) properties of quadratic forms. The objective of the present note is twofold: In Sections 2–4 we extend the structure theory of such fields and, moreover, generalize some of their Galois-theoretic properties into a purely group-theoretic setting. Next we deal with the following (known) structure theorem: $G_{K(\sqrt{-1})}(2)$ is a free pro-2 group if and only if $G = G_K(2)$ is a free pro-2 product (in a natural sense) of groups of order 2 and of a free pro-2 group. This deep fact has been proven by Eršov [Er] and Ware [W3] using field-theoretic tools. It was generalized by Haran [H4] to an arbitrary pro-2 group $G$ (under a mild assumption arising from Artin–Schreier theory). The proof in [H4], however, uses heavy machinery: a cohomology theory for the category of the so-called Artin–Schreier structures, and the study of projective resolutions of profinite $G$-modules (these tools are also partly developed in [H2] and [H3]). Our second goal is thus to give a simplified proof of this fundamental fact, using only standard methods of *Galois* cohomology. This is done in Section 5, using the results of the previous sections.

Our approach is to explore the cohomological connections between a profinite group $G$ and its *real core* $N$; that is, $N$ is the closed subgroup generated by all the involutions in $G$. For $G$ as above (or, more generally, when $G$ is *virtually projective of real type;* cf. Sections 2–3) we obtain a short exact sequence relating $N$ to the Bockstein operator of $G$ (Corollary 3.4). Combined with an approximation property for $H^1(G, \mathbb{Z}/2\mathbb{Z})$, this is used to show that $G/N$ is projective—that is, has cohomological dimension $\leq 1$ (Theorem 4.5; see also Remark 5.3(2)).

This latter fact is of particular interest in studying the structure of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. Indeed, denote the field of totally real algebraic numbers by $\mathbb{Q}_{tr}$, let $\mathbb{Q}_{ab}$ be the maximal pro-abelian extension of $\mathbb{Q}$, and let $\bar{\mathbb{Q}}_{ab} = \mathbb{Q}_{ab} \cap \mathbb{R} = \mathbb{Q}_{ab} \cap \mathbb{Q}_{tr}$. Our results then imply that $\mathrm{Gal}(\mathbb{Q}_{tr}/\bar{\mathbb{Q}}_{ab})$

Received May 16, 1994. Revision received May 18, 1995.
Michigan Math. J. 42 (1995).