# The Branched Schwarz Lemma: A Classical Result via Circle Packing

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## 1. Introduction

Our original aim in this work was to establish certain results about circle packings which were suggested by classical complex function theory. We succeeded in doing that, but along the way we found that we could in fact extend the classical results themselves, independently establishing a theorem of Z. Nehari.

A circle packing is a collection of circles in the plane with a prescribed pattern of tangencies satisfying certain combinatoric conditions, which will be described in a moment. Connections between circle packings and analytic functions were introduced by W. Thurston in 1985. The seminal paper in this topic is the proof by Rodin and Sullivan [RS] of Thurston's conjecture on the approximation of conformal mappings via circle packings. Subsequent work by several researchers has refined the approximation results, but has also suggested the possibility of developing a "discrete analytic function" theory based on circle packings which would parallel classical analytic function theory. Here, a thorough mixing of the proofs of certain fundamental classical, discrete, and approximation results suggests that the emerging discrete theory provides far more than mere analogy with its classical counterpart.

It is certainly best for the reader if we state the classical results first. A finite Blaschke product is an $n$-to-1 proper mapping of the unit disc $\mathbf{D}$ onto itself for some positive integer $n$ and $\mathrm{br}(f)$ denotes the set of branch points of an analytic function $f$, counting multiplicities.

Schwarz's Lemma (Branched). *Let $f, b\colon \mathbf{D} \to \mathbf{D}$ be analytic, with $f(0) = 0 = b(0)$, and assume $b$ is a finite Blaschke product. If $\mathrm{br}(b) \subseteq \mathrm{br}(f)$, counting multiplicities, then $|f'(0)| \leq |b'(0)|$. If $|b'(0)| \neq 0$, then equality holds iff $f \equiv \lambda b$ for some unimodular constant $\lambda$.*

Distortion Lemma (Branched). *Let $f\colon \mathbf{D} \to \mathbf{C}$ be analytic with $f(0) = 0$ and let $r > 0$. Write $\Omega$ for the component of $f^{-1}(r\mathbf{D})$ containing $0$ and assume that the restriction $f|_{\Omega}\colon \Omega \to r\mathbf{D}$ is a proper mapping. Let $b$ be a finite*

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