On the Equivalence of Holomorphic and Plurisubharmonic Phragmén–Lindelöf Principles

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There are several papers in recent years which studied partial differential operators $P(D)$ on classes of infinitely differentiable functions on convex open sets in $\mathbb{R}^N$ or $\mathbb{C}^N$ in terms of Phragmén–Lindelöf type estimates for plurisubharmonic functions on algebraic varieties. In the early work of Hörmander [7] it is shown that the surjectivity of $P(D)$ on the space of all real analytic functions on a convex open set in $\mathbb{R}^N$ is equivalent to Phragmén-Lindelöf conditions on the tangent cone at infinity of the variety $V(P) := \{z \in \mathbb{C}^N \mid P(z) = 0\}$. Later Zampieri [13], Braun, Meise, and Vogt [3; 4], and Braun [1] made similar investigations for classes of ultradifferentiable functions. Kaneko [8] proved that Hartogs problems for partial differential operators $P(D)$ can be characterized by Phragmén–Lindelöf conditions on $V(P)$. To treat the problem of existence of continuous linear right inverses for partial differential operators, Meise, Taylor, and Vogt [9], Momm [11], and Palamodov [12] also used Phragmén–Lindelöf conditions. In most of the aforementioned cases one first derives the Phragmén–Lindelöf conditions for all plurisubharmonic functions $u = \log|f|$, where f is a holomorphic function on $V(P)$. Meise, Taylor, and Vogt [10] proved a general result which shows that the conditions for all plurisubharmonic functions of type $u = \log|f|$, where f is a holomorphic function on $V(P)$, hold if and only if the conditions hold for all plurisubharmonic functions on $V(P)$. The idea is to write the plurisubharmonic function u as an upper envelope of functions $\log|f|$. More precisely, they have shown that for each $0 < \theta < 1$ and for each plurisubharmonic function u on the variety $V(P)$ with $u(z) \le |z|$, and for most of the regular points $\zeta \in V_{\mathrm{reg}}(P)$, there exists a holomorphic function f on $V(P)$ such that

$$\log|f(z)| \le \sup\{u(y) \mid |z-y| \le 1\} + C\log(2+|z|), \quad z \in V(P)$$

and

$$\log|f(\zeta)| \ge \theta u(\zeta) - C\log(2+|\zeta|),$$

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