Siegel's Lemma for Function Fields

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Introduction

Since the work of Thue early this century, an important tool in transcendental number theory and Diophantine approximation is the fact that a system of homogeneous linear equations over $\mathbb{Q}$ has a relatively small integer solution. This idea was formalized by Siegel in 1929 [7], and it has since been common to refer to results along this line of thought as "Siegel's lemma". Using the notion of a height, one can formulate the question of finding small solutions to systems of linear equations for arbitrary global fields—that is, fields for which one has a "product formula". It has proven useful to give versions of Siegel's lemma for such fields other than $\mathbb{Q}$. This was carried out for number fields by Bombieri and Vaaler in [3]. Here we formulate and prove a Siegel's lemma for function fields, where by "function field" we mean any finite algebraic extension of a field of rational functions in one indeterminate. We will give definitions for the heights used below in the next section.

Throughout this paper, $k = k_0(T)$ will denote the field of rational functions in one indeterminate over the field k_0 (we put no restrictions on the field k_0). We let K be any finite algebraic extension of k. As in [2], we denote by K_0 the *field of constants* of K (this is the algebraic closure of k_0 in K by [2, Chap. 12, Thm. 6]) and the *effective degree* by $\mathfrak{m}(K,k) = [K:k]/[K_0:k_0]$. More generally, if L is a finite algebraic extension of K with field of constants L_0, then the effective degree of this extension is $\mathfrak{m}(L,K) = [L:K]/[L_0:K_0]$.

THEOREM 1. *Let K be a function field and let h_A be a height on K^n (as defined below). There is a basis $\mathbf{a}_1, \dots, \mathbf{a}_n$ of K^n satisfying*

$$\sum_{i=1}^{n} h_A(\mathbf{a}_i) \le h_A(K^n) + \frac{n}{\mathfrak{m}(K,k)}(g-1+\mathfrak{m}(K,k)),$$

where g is the genus of K.

As will be shown in Lemma 5, we always have the lower bound for a basis

$$\sum_{i=1}^{n} h_A(\mathbf{a}_i) \ge h_A(K^n),$$

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