# Essentially Normal Multiplication Operators on the Dirichlet Space

JAROSŁAW LECH

## 1. Introduction

Let $U$ be the open unit disk in the complex plane $\mathbf{C}$. The Dirichlet space $D$ is the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $U$ such that

$$f(0) = 0 \quad \text{and} \quad \|f\|_D^2 = \int_U |f'(z)|^2 \frac{dA}{\pi} = \sum_{n=1}^{\infty} n|a_n|^2 < \infty,$$

where $dA$ denotes the usual area measure.

An analytic function $\varphi$ on $U$ is called a *multiplier* of $D$ if $\varphi D \subset D$. The set of all multipliers of $D$ will be denoted by $M(D)$. Each multiplier generates a bounded multiplication operator $M_\varphi$ on $D$ defined by $M_\varphi f = \varphi f$ for $f \in D$.

Multiplication operators on $D$ are almost never normal (they are normal only for constant multipliers). In [AS], Axler and Shields asked whether the self-commutator $M_\varphi^* M_\varphi - M_\varphi M_\varphi^*$ is compact for $\varphi \in M(D)$; that is, whether multiplication operators on $D$ are normal in the Calkin algebra. A Hilbert space operator whose self-commutator is compact is called *essentially normal*.

This paper answers negatively the question of Axler and Shields. An example of a multiplication operator that is not essentially normal is given in Section 3. Section 2 contains a description of essentially normal multipliers that is used throughout the rest of the paper.

A few more definitions are in order. The *harmonic* Dirichlet space $D_h$ is the Hilbert space of functions $f$ on the unit circle $T$ for which

$$\|f\|_{D_h}^2 = |\hat{f}(0)|^2 + \sum_{n=-\infty}^{\infty} |n||\hat{f}(n)|^2 < \infty,$$

where $(\hat{f}(n))$ is the sequence of Fourier coefficients of $f$. It can be shown that

$$\begin{aligned}\|f\|_{D_h}^2 &= |\hat{f}(0)|^2 + \int_U |\nabla P[f]|^2 \frac{dA}{\pi} \\ &= |\hat{f}(0)|^2 + \int_0^{2\pi}\int_0^{2\pi} \left| \frac{f(e^{i\theta}) - f(e^{i\xi})}{e^{i\theta} - e^{i\xi}} \right|^2 \frac{d\theta}{2\pi}\frac{d\xi}{2\pi},\end{aligned}$$

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