# A Characterization of Virtual Poincaré Duality Groups

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An important open question in geometric group theory and the topology of manifolds asks whether every Poincaré duality group is realized as the fundamental group of a closed aspherical manifold. This note establishes an analogous property for virtual Poincaré duality groups following ideas and results of [10; 11; 12] and gives an approximate solution to the realization question for Poincaré duality groups.

We recall that a group $\Gamma$ is a Poincaré duality group if $\Gamma$ is of type FP and, for some $n>0$, $H^i(\Gamma;\mathbf{Z}\Gamma)=0$ for $i\neq n$ and $H^n(\Gamma;\mathbf{Z}\Gamma)$ is an infinite cyclic group [1; 7; 4]. (For finitely presented groups, these conditions are equivalent to the topological condition that the Eilenberg–MacLane space $B\Gamma$ is a Poincaré complex.) Such a group satisfies Poincaré duality in the form

$$H^i(\Gamma;M)\cong H_{n-i}(\Gamma;\tilde{M}),$$

where $M$ is a $\mathbf{Z}\Gamma$ module, $\tilde{M}=H^n(\Gamma;\mathbf{Z}\Gamma)\otimes_{\mathbf{Z}}M$, and the tensor product is given the diagonal $\Gamma$-action. (The infinite cyclic group $D=H^n(\Gamma;\mathbf{Z}\Gamma)$ is customarily called the *dualizing module* and has an associated orientation character $w:\Gamma\to\{1,-1\}$ such that $\gamma\cdot x=w(\gamma)x$ for each $\gamma\in\Gamma$ and each $x\in D$; $\tilde{M}$ is therefore described as "$M$ twisted by the orientation character".)

Recall also that a group $G$ is said to have a *virtual* property if and only if $G$ possesses a finite-index subgroup $H$ with that property. Thus, a group $G$ is of finite virtual cohomological dimension if some subgroup $H$ of finite index in $G$ has finite cohomological dimension, and $G$ is a virtual Poincaré duality group if some finite-index subgroup $H$ is a Poincaré duality group.

The fundamental group of any closed aspherical manifold is a Poincaré duality group, and every known Poincaré duality group is of this form. Since many virtual Poincaré duality groups contain elements of finite order, we can not expect such groups to be realized as fundamental groups of aspherical manifolds.

THEOREM 1. *Let $\Gamma$ be a finitely presented group of finite virtual cohomological dimension. Then $\Gamma$ is a virtual Poincaré duality group if and only if there exists a closed PL manifold $M$ with fundamental group $\Gamma$ and universal cover $\tilde{M}$ such that $\tilde{M}$ is homotopy equivalent to a finite complex.*

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