Point Evaluations for $P^t(\mu)$ and the Boundary of Support(μ)

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1. Introduction

Let G be a bounded, simply connected region in the complex plane $\mathbf{C}$, and for $1 \le t < \infty$ let μ be a finite, positive Borel measure with support in $\bar{G}$ such that each function in $P^t(\mu)$, the closure of the polynomials in $L^t(\mu)$, has an analytic continuation to G. When can we be assured that there exists λ in ∂G and $r > 0$ such that any function in $P^t(\mu)$ has an analytic continuation beyond G to $G \cup \{z : |z-\lambda| < r\}$? J. Brennan has answered this for many weighted area measures (see [B2, Thm. 4]). In this paper we examine this question and others like it with very few, if any, restrictions on μ, though we are, almost of necessity, more specific about G. As one might expect, J. Thomson's recent theorem on point evaluations (see [Th, Thm. 5.8] or Theorem 2.1 in this paper) is useful to us here.

If $1 \le t < \infty$ and μ is a finite, positive Borel measure with compact support in $\mathbf{C}$, then abpe$(P^t(\mu))$ denotes the collection of analytic bounded evaluations for the polynomials with respect to the $L^t(\mu)$ norm, and is the largest open subset of $\mathbf{C}$ to which every function in $P^t(\mu)$ has an analytic continuation. Let G be a bounded, simply connected region and let μ have support in $\bar{G}$ such that $G \subseteq \text{abpe}(P^t(\mu))$. Among such measures are those of the form $d\mu = w\, dm_2|_G + h\, d\omega_G$, where m_2 is a 2-dimensional Lebesgue measure on $\mathbf{C}$, $\omega_G := \omega(\cdot, G, z_0)$ is harmonic measure on ∂G evaluated at some z_0 in G, $0 \le w \in L^1(m_2|_G)$, $0 \le h \in L^1(\omega_G)$, and either w is positive and continuous or $\int \log(h)\, d\omega_G > -\infty$. If $\text{Rat}(\bar{G}) \nsubseteq P^t(\mu)$, where $\text{Rat}(\bar{G})$ is the collection of rational functions with poles off the closure of G, then $\mathbf{C} \setminus \bar{G}$ has bounded components and at least one of them, along with G, is contained in abpe$(P^t(\mu))$. Under these circumstances, if $\mu|_{\partial G}$ is small enough, then [Th, Thm. 5.8] implies that $(\partial G) \cap \text{abpe}(P^t(\mu)) \neq \emptyset$. However, the size of $\mu|_{\partial G}$ does not alone determine whether or not ∂G meets abpe$(P^t(\mu))$. For instance, there are measures μ_1 and μ_2 having the properties of μ described earlier such that $\text{Rat}(\bar{G}) \nsubseteq P^t(\mu_i)$ $(i = 1, 2)$, $\mu_1|_{\partial G} = \mu_2|_{\partial G}$, and $(\partial G) \cap \text{abpe}(P^t(\mu_1)) = \emptyset$; in fact,

$$[\text{abpe}(P^t(\mu_1))] \setminus (\partial G) = [\text{abpe}(P^t(\mu_2))] \setminus (\partial G)$$

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