Holomorphic Mappings, the Schwarz–Pick Lemma, and Curvature

DAVID GOLOFF & WING-KEUNG TO

0. Introduction

The main result of this paper bounds the dimension of the complex space of rank-k holomorphic mappings between two compact complex manifolds X and Y, which we denote by $\mathrm{Hol}_k(X,Y)$. The hypothesis of the main theorem involves curvature conditions on the image manifold Y. There are also two lemmas of independent interest. One shows that the evaluation mapping at any point $x\in X$, which we denote by $\mathrm{eval}(x)\colon \mathrm{Hol}_k(X,Y)\to Y$, does not reduce dimension. The other lemma is a variation on the Schwarz–Pick lemma. These results are part of the school of thought exemplified in [KSW], [K1], [NS], [No], [SY], and [La].

We list some results that motivate this work as follows.

0. (De Franchis' theorem) If X and Y are compact 1-dimensional complex manifolds (i.e. Riemann surfaces) and the genus of Y is at least 2, then $\dim \mathrm{Hol}_1(X,Y)=0$.
1. [KSW] If X and Y are complex manifolds, X is compact, and the holomorphic tangent bundle $T(Y)$ has a Hermitian metric such that the curvature form $R(v,\bar v)(\ ,\ )$ has at least $n-k$ negative eigenvalues for each nonzero vector $v\in T(Y)$, then $\dim \mathrm{Hol}_{k+1}(X,Y)=0$.

   Here, $R(v,\bar w)(s,\bar t)=h(D^2(s,\bar t)v,\bar w)$, where D is the connection associated to the metric h (see [GA] or [SS]).
2. [K2] Let M be a compact complex manifold whose first Chern class $c_1(M)$ is represented by a $(1,1)$-form

$$\frac{i}{2\pi}\sum_{a,b} C_{a\bar b}\,dz_a\wedge d\bar z_b.$$

   If $\{C_{a\bar b}\}$ is negative semidefinite with maximal rank r, then

$$\dim \mathrm{Aut}(M)\le \dim M-r,$$

   where $\mathrm{Aut}(M)$ denotes the automorphism group of M.
3. [NS] Let X be a compact complex manifold. If $\bigwedge^k T(Y)$ has a Hermitian metric such that $R(v,\bar v)(\ ,\ )$ is negative definite for each nonzero $v\in\bigwedge^k T(Y)$, then $\dim \mathrm{Mero}_k(X,Y)=0$. Here, "$\mathrm{Mero}_k$" stands for

Received August 27, 1991. Revision received September 20, 1994.
Michigan Math. J. 42 (1995).