Soul-Preserving Submersions

GERARD WALSCHAP

In this note, we investigate the structure of Riemannian submersions π: $M \to N$ from open manifolds M with nonnegative sectional curvature K. By O'Neill's formula [5], N also has nonnegative curvature. In fact, most—but not all, see Question 2.4—open manifolds N with $K \geq 0$ arise in this fashion, by taking M to be a Riemannian product $M' \times P^k$, where P^k is diffeomorphic to $\mathbf{R}^k$.

The starting point is the observation that if N is open, then the pre-image of a soul is a totally convex submanifold M' of M. It follows that if π has compact fibers, then it is soul-preserving. Moreover, the structure of π is essentially determined by its restriction to the tangent and normal bundles of the soul Σ_M of M. This is used to derive a classification of the metric fibrations of $S^n \times \mathbf{R}^k$ with compact fibers, which turn out to be homogeneous —that is, generated by the action of a group of isometries—if $n \neq 15$. The curvature of the base is actually indicative of the metric structure of open manifolds with $K \geq 0$ in general: topologically, the base is a nontrivial vector bundle over the soul with positively curved fibers. This is true for any complete, noncompact M with $K \geq 0$: If every plane orthogonal to Σ_M has zero curvature, then M splits as a metric product $\Sigma_M \times P^l$, at least locally.

We mention two further applications. The first is that positively curved open manifolds admit no metric fibrations. The second is that 1-dimensional Riemannian fibrations of locally symmetric (open) spaces with $K \geq 0$ are homogeneous, unless perhaps the quotient space is trivial in the sense that it is isometric to a product of a compact manifold with Euclidean space.

1. Convexity and Submersions

The reader is referred to [1] for facts about open manifolds of nonnegative curvature that will be used freely. We adopt the notation of [4] for the basic geometric invariants of Riemannian foliations. Thus, the foliation $\mathfrak{F}$ on the (complete) manifold M determines an orthogonal splitting $TM = \Delta^h \oplus \Delta^v$ of the tangent bundle into so-called horizontal and vertical subbundles, where

Received February 3, 1994. Revision received July 1, 1994.
Supported in part by a grant from the National Science Foundation.
Michigan Math. J. 41 (1994).