The Dirichlet Problem for the Complex Monge–Ampère Operator: Perron Classes and Rotation-Invariant Measures

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0. Introduction

Let Ω be an open and bounded subset of $\mathbb{C}^n$. If $u_j \in C^2(\Omega)$, $1 \le j \le n$, then the Monge–Ampère operator $(dd^c)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n$ operates on $(u_1, \dots, u_n)$, where $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. This operator is of great importance in pluripotential theory. It was shown in [3] that $(dd^c)^n$ is well-defined and nonnegative on $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}$. In this paper, we will study the following Dirichlet problem.

Let Ω be an open, bounded, and strictly pseudoconvex subset of $\mathbb{C}^n$, let φ be in $C(\partial\Omega)$, and let μ be a positive measure on Ω. Consider the problem:

$$\begin{cases} u \in \mathrm{PSH} \cap L^\infty(\Omega), \\ (dd^c u)^n = \mu \ \text{ on } \Omega \\ \overline{\lim}_{z \to \xi} u(z) = \varphi(\xi) \ \ \forall \xi \in \partial\Omega. \end{cases} \tag{i}$$

There are measures for which (i) has no solution. For if (i) can be solved with μ, then μ cannot have mass on any pluripolar set. Thus, for example, if we take μ to be the Dirac measure for a point in Ω then (i) has no solution. On the other hand, if $\mu = f\,dV$ where $f \in C(\bar\Omega)$ and dV is Lebesgue measure, then it was proved in [2] that (i) has a unique solution for every $\varphi \in C(\partial\Omega)$. This was generalized in [5] to the case when $f \in L^\infty(\Omega, \mu)$ and in [7] to the case when $f \in L^2(\Omega, dV)$. The main result of this paper is the following. Let ν be any positive rotation invariant measure in the unit ball B for which there is a $u \in \mathrm{PSH} \cap L^\infty(B)$ with $(dd^c u)^n \ge \nu$. (These measures can be characterized; cf. [11].) Then, for every $f \in L^\infty(B, \nu)$ and for every $\varphi \in C(\partial B)$, there is a unique solution to (i) with $\mu = f\,d\nu$. For background and references see [1; 6; 10].

1. Perron Classes

To study the problem (i), we use the Perron method and therefore consider classes of subsolutions

Received October 6, 1993. Revision received January 28, 1994.
[1] Partially supported by the Swedish Natural Science Research Council.
[2] Partially supported by grant no. 2 1077 91 from the State Committee for Scientific Research (Poland).
Michigan Math. J. 41 (1994).