On the Boundary Behavior of Singular Inner Functions

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1. Introduction

A holomorphic function f in the open unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is called an *inner function* if $|f(z)| \leq 1$ for $z \in \mathbf{D}$ and if f has radial limits $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ of modulus 1 at almost every point $e^{i\theta}$ of the unit circle $\mathbf{T} = \partial\mathbf{D}$. If (a_k) is a sequence of complex numbers in $\mathbf{D}$ that satisfies the Blaschke condition $\sum_k (1 - |a_k|) < \infty$, then the *Blaschke product*

$$B(z) = \prod_k \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbf{D},$$

is an inner function whose zeros are exactly at the points (a_k). The boundary behavior of Blaschke products was investigated in various contexts. For a survey on this subject we refer to Colwell [7, pp. 13–44, 83]. In particular, Belna, Carroll, Colwell, and Piranian have shown in [2] and [3] that the radial boundary behavior of Blaschke products can be prescribed on any countable subset E of $\mathbf{T}$. In [11] and [12] Nicolau extended some of their results to even more general subsets E of $\mathbf{T}$ of Lebesgue measure zero.

Contrary to the situation for Blaschke products, related questions on the boundary behavior of the second basic type of inner functions, the *singular* inner functions, remained open. If f is an inner function that does not vanish in $\mathbf{D}$, then $f = cS_\mu$, where c is a unimodular constant and

$$S_\mu(z) = \exp\left(-\int_{\mathbf{T}} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t)\right), \quad z \in \mathbf{D},$$

is given by a (uniquely determined) positive finite Borel measure μ on T which is singular with respect to linear Lebesgue measure. Therefore S_μ is called a *singular inner function* with associated measure μ (cf. [13, pp. 32–33). For simplicity we shall often identify $e^{it} \in \mathbf{T}$ with $t \in (-\pi, \pi]$.

It is the aim of this paper to establish the existence of singular inner functions having a prescribed radial boundary behavior. Here, in contrast to Blaschke products, we shall be concerned with *two* different kinds of nonvanishing inner functions. For *discrete* singular inner functions S_σ, the generating

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