Quasiextremal Distance Domains and Integrability of Derivatives of Conformal Mappings

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1. Introduction

Let Ω be a simply connected domain in the complex plane and let f be a conformal mapping from Ω onto the unit disk Δ. Let L be any straight line in the plane. In this note we consider the following question: For which values of p is

$$\int_{L\cap\Omega} |f'(z)|^p\,ds < \infty?$$

That is, for which values of p is it true that $f' \in L^p(L\cap\Omega)$? Without loss of generality, the straight line L may be taken to be the real axis R. For $p=1$, Hayman and Wu [HW] proved that this integral is bounded by a universal constant (see also [GGJ], [FHM], and [Øy]). The Koebe function shows that $f' \in L^2(R\cap\Omega)$ can fail. In [Ha, p. 638], Baernstein conjectured that $f' \in L^p(R\cap\Omega)$ would be true for all $p \in [1,2)$. In the positive direction of this conjecture, Fernandez, Heinonen, and Martio [FHM] proved that there is an absolute constant $\epsilon > 0$ such that $f' \in L^p(R\cap\Omega)$ for all $p \in [1, 1+\epsilon)$. But in [Ba] Baernstein gave a beautiful counterexample to the conjecture. In this paper we give several sufficient conditions which ensure that $f' \in L^p(R\cap\Omega)$ for all $p \in [1,2)$.

In Section 2 we consider quasiextremal distance (or QED) domains and obtain some sharp estimates for the QED constants of certain domains. These estimates have their own interest, but here they are used to prove one of our main results on Baernstein's disproven conjecture. In Sections 3 and 4 we give several geometric conditions, some on $f(L)$ and some on Ω, which ensure that $f' \in L^p(R\cap\Omega)$ for all $p \in [1,2)$. For example, in Section 3 we show that if $f(L)$ is a Jordan curve of bounded rotation, then $f' \in L^p(L\cap\Omega)$ for all $p \in [1,2)$. We also show that if Ω is starlike, then $f' \in L^p(L\cap\Omega)$ for all $p \in [1,2)$.

By modifying Baernstein's example slightly, we construct in Section 5 a chord-arc domain Ω such that $R\cap\partial\Omega$ contains a single point and that $f' \notin L^p(R\cap\Omega)$ for some $p \in (1,2)$. This example suggests that severe conditions

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