# A Characterization of Domains Quasiconformally Equivalent to the Unit Ball

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A difficult problem in the theory of quasiconformal mappings is the characterization of domains in Euclidean $n$-space that can be mapped quasiconformally onto the unit ball $\mathbf{B}^n$. It is a type of $n$-dimensional Riemann mapping problem. Gehring [Ge] reduced the problem for a domain to a problem for its boundary. He showed that if a quasiconformal mapping exists from a neighborhood of a domain's boundary onto a neighborhood in $\mathbf{B}^n$ of $\mathbf{S}^{n-1}$, then a quasiconformal mapping exists between the domain and $\mathbf{B}^n$. But unlike the Riemann mapping theorem (which solves the problem for $n=2$), no conditions pertaining solely to the boundary have been discovered which guarantee a domain's quasiconformal equivalence to $\mathbf{B}^n$.

Nakai [N1] established an implicit characterization of quasiconformally equivalent domains in terms of function algebras by showing that a quasiconformal mapping exists between two domains in $\mathbf{R}^2$ if and only if their corresponding Royden algebras are isomorphic. Lewis [Le] and Lelong-Ferrand [L-F] extended his proof to higher dimensions and Riemannian manifolds. The work of Nakai and Lewis relies on methods of functional analysis to characterize the maximal ideal space of a domain $\Omega$ as a compactification $\Omega^*$ of $\Omega$. Lewis showed that a quasiconformal mapping between two domains implies that these so-called *Royden compactifications* are homeomorphic. The converse question—whether a homeomorphism between Royden compactifications implies existence of a quasiconformal mapping between domains—is the subject of this paper.

Nakai [N2] answered the question affirmatively for the 2-dimensional case, and gave a partial answer for higher dimensions by showing [NT] that the restriction of a homeomorphism between Royden compactifications to their underlying domains is quasiconformal in a neighborhood of the boundary. A connection between this result and the theorem of Gehring has not been previously observed in the literature. However, if one of the domains is $\mathbf{B}^n$, then the result of Nakai and Tanaka dovetails nicely with Gehring's theorem to prove that the converse question can be answered affirmatively. Hence, a domain $\Omega$ is quasiconformally equivalent to $\mathbf{B}^n$ if and only if $\Omega^*$ is

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