Unimodular Wavelets for L^2 and the Hardy Space H^2

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1. Introduction

In this paper we construct a family of wavelets ψ for L^2 and the Hardy space H^2 with the property that $|\hat{\psi}(\xi)|=1$ for ξ in the support of $\hat{\psi}$. One of the wavelets constructed is the well-known Journé–Meyer example. We also include a proof of the equivalence of Meyer's equations and wavelet conditions.

Let $\psi \in L^2(\mathbf{R})$ and let, for $j, k \in \mathbf{Z}$,

$$\psi_{j,k}(x)=2^{j/2}\psi(2^j x-k).$$

Let H be a Hilbert subspace of $L^2(\mathbf{R})$. The function $\psi \in H$ is called a *wavelet* for H if $\{\psi_{j,k}\}_{j,k\in\mathbf{Z}}$ forms an orthonormal basis for H; $\{\psi_{j,k}\}$ is called a *wavelet basis.*

There are essentially two methods of constructing wavelets. The first is based on the following equations (W1)–(W4): $\psi \in L^2$ is a wavelet for $L^2(\mathbf{R})$ if and only if ψ satisfies

(W1) $\sum_{k\in\mathbf{Z}}|\hat{\psi}(\xi+2k\pi)|^2=1$,
(W2) $\sum_{k\in\mathbf{Z}}\hat{\psi}(\xi+2k\pi)\hat{\psi}^*(2^j(\xi+2k\pi))=0$ for $j\geq 1$,
(W3) $\sum_{j\in\mathbf{Z}}|\hat{\psi}(2^{-j}\xi)|^2=1$, and
(W4) $\sum_{l\geq 0}\hat{\psi}(2^l(\xi+2p_0\pi))\hat{\psi}^*(2^l\xi)=0$ for $p_0\in 2\mathbf{Z}+1$.

Here, $\hat{\psi}^*$ is the complex conjugate of $\hat{\psi}$. This equivalence appears in [Le] and is attributed to Y. Meyer. However, no proof of it seems available, so we give a complete proof in this paper.

The second method of constructing wavelets is based on the pairing of wavelets and multiresolution analysis (MRA) [Me, D2]. An increasing sequence $\{V_j\}$ of closed subspaces of $L^2(\mathbf{R})$ is called an *MRA* of $L^2(\mathbf{R})$ if the following hold:

(R1) $\bigcap_{j\in\mathbf{Z}}V_j=\{0\}$ and $\overline{\bigcup_{j\in\mathbf{Z}}V_j}=L^2(\mathbf{R})$,
(R2) $f(x)\in V_j$ if and only if $f(2^{-j}x)\in V_0$,

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