Bourgain Algebras of Spaces of Harmonic Functions

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1. Introduction

Bourgain algebras were introduced by Cima and Timoney [5] in connection with Bourgain's work on the Dunford–Pettis property for certain concrete function algebras [1]. Subsequently, several authors have studied Bourgain algebras [2; 3; 4; 7; 8; 9; 12; 14; 15] with a variety of goals in mind but with attention to the Bourgain algebra determined by an *algebra.* We are concerned with the study of Bourgain algebras of a class, including the space of bounded harmonic functions on the disk, of *linear subspaces.*

Let $\mathbf{D}$ be the open unit disk in the complex plane $\mathbf{C}$ and let $\mathbf{T}=\partial\mathbf{D}$ be the unit circle. The usual spaces of essentially bounded functions with respect to Lebesgue measure are denoted by $L^\infty(\mathbf{T})$ and $L^\infty(\mathbf{D})$. The space of bounded analytic functions on $\mathbf{D}$ is denoted by $H^\infty(\mathbf{D})$, with $H^\infty=H^\infty(\mathbf{T})$ being used to denote the boundary values of $H^\infty(\mathbf{D})$ functions. We will also write $L^\infty=L^\infty(\mathbf{T})$ for brevity. The algebra of bounded continuous functions on $\mathbf{D}$ is denoted by $BC(\mathbf{D})$ and $C=C(\mathbf{T})$ denotes the algebra of continuous functions on $\mathbf{T}$. Each of these algebras is equipped with the (essential) supremum norm $\|\ \|_\infty$.

Let $\mathcal{X}$ be one of the spaces L^∞, $L^\infty(\mathbf{D})$ or $BC(\mathbf{D})$, and let $\mathcal{Y}\subset\mathcal{X}$ be a closed linear subspace. We say that $f\in\mathcal{X}$ belongs to the *Bourgain algebra of* $\mathcal{Y}$ *relative to* $\mathcal{X}$, and write $f\in\mathcal{Y}_b$, in case for every weakly null sequence $\{f_n\}$ in $\mathcal{Y}$ there exists a sequence $\{g_n\}$ in $\mathcal{Y}$ such that $\|f_nf-g_n\|_\infty\to 0$ as $n\to\infty$. Essentially as shown in [5], $\mathcal{Y}_b$ contains the constants and is a closed subalgebra of $\mathcal{X}$. Moreover, if $\mathcal{Y}$ is a *subalgebra,* then $\mathcal{Y}\subset\mathcal{Y}_b$. However, there is no known simple relationship between a *subspace* $\mathcal{Y}$ and its Bourgain algebra $\mathcal{Y}_b$. We emphasize that $\mathcal{Y}_b$ is defined relative to a particular overlying space $\mathcal{X}$ even though this is not reflected in the notation $\mathcal{Y}_b$. Each of the spaces $\mathcal{X}=L^\infty$, $L^\infty(\mathbf{D})$, $BC(\mathbf{D})$, or $C(\mathfrak{M})$ (when $\mathcal{Y}$ is an algebra with maximal ideal space $\mathfrak{M}$) has a certain claim to naturality; however, the general dependence of $\mathcal{Y}_b$ upon $\mathcal{X}$ is quite complicated and not fully understood. For $\mathcal{X}=L^\infty(\mathbf{D})$ and $\mathcal{Y}$ a subalgebra containing the bounded analytic functions,

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