Complements of Runge Domains and Holomorphic Hulls

FRANC FORSTNERIC

0. Introduction

A compact subset A of a Stein manifold X is said to be *holomorphically convex* if for every $z \in X \setminus A$ there is a holomorphic function f on X such that $|f(z)| > \sup_A |f|$. The holomorphic hull $\hat{A}$ of a compact set $A \subset X$ is the smallest holomorphically convex set containing A, given by

$$\hat{A} = \{z \in X : |f(z)| \leq \sup_A |f| \text{ for all } f \in \mathcal{O}(X)\}.$$

When $X = \mathbf{C}^n$, $\hat{A}$ is the *polynomially* convex hull of A.

Holomorphically convex sets play an extremely important role in complex analysis; we refer the reader to Hörmander [7]. It is usually a difficult problem to decide whether a given set is holomorphically convex, or to determine its holomorphic hull. Therefore, the results that provide obstructions to holomorphic convexity, or that give estimates on the size and shape of the hull, are of interest to complex analysts.

Recently Alexander [1] applied the notion of *linking* to this problem. If a compact set $K \subset \mathbf{C}^n$ is linked by an orientable closed manifold $Y \subset \mathbf{C}^n \setminus K$ of real dimension $q \leq n-1$, in the sense that Y is not homologous to zero in $\mathbf{C}^n \setminus K$, then Y must intersect the polynomial hull $\hat{K}$ [1, Thm. 1]. Alexander established a similar result for sets in Stein manifolds. His proof uses differential forms (via de Rham's theorem) and the Poincaré duality. This approach necessitates the use of homology and cohomology with real (or complex) coefficients. Specifically, the assumption that Y is not homologous to zero in $\mathbf{C}^n \setminus K$ means that there exists a differential q-form ω on $\mathbf{C}^n \setminus K$ such that $\int_Y \omega \neq 0$.

In the present paper we obtain more general topological properties of complements of holomorphically convex subsets $A \subset X$ in Stein manifolds X of dimension $n \geq 2$. We show that the inclusion $X \setminus A \hookrightarrow X$ induces isomorphism of low dimensional homology groups (up to dimension $n-2$), with coefficients in an arbitrary abelian coefficient group G. If X is contractible, the groups $\mathrm{H}_k(X \setminus A; G)$ vanish in dimensions $1 \leq k \leq n-1$. When $X = \mathbf{C}^n$, the homotopy groups of $\mathbf{C}^n \setminus A$ in the same dimensions vanish as

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