A Fibered Polynomial Hull without an Analytic Selection

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This note shows that certain polynomial hulls in $\mathbf{C}^3$ have no analytic selection, thus settling a standing question about such hulls.

Recall that the *polynomial hull* $\mathcal{P}(\mathcal{S})$ of a set $\mathcal{S}$ in $\mathbf{C}^M$ is the set

$$\mathcal{P}(\mathcal{S})=\{w\in\mathbf{C}^M : |p(w)|\le\max_{v\in\mathcal{S}}|p(v)| \text{ for all polynomials } p \text{ on } \mathbf{C}^M\}.$$

We shall be considering sets $\mathcal{S}$ in $\mathbf{C}^{1+N}$ that are fibered over the unit circle $\partial\mathbf{D}$ in $\mathbf{C}$ the complex plane. Thus $\mathcal{S}$ has the form

$$\{(e^{i\theta},\mathcal{S}_\theta):0\le\theta\le 2\pi\},$$

where each $\mathcal{S}_\theta$ is a subset of $\mathbf{C}^N$. Let $\mathbf{D}$ denote the unit disk in the complex plane and let H_N^∞ denote the $\mathbf{C}^N$-valued functions bounded and analytic on $\mathbf{D}$. Clearly (from the maximum principle) if f is any H_N^∞ function satisfying

$$f(e^{i\theta})\in\mathcal{S}_\theta \text{ for almost all } \theta,$$

then the graph $\{z,f(z)):z\in\mathbf{D}\}$ of f lies in $\mathcal{P}(\mathcal{S})$. Such a function f is called an *analytic selection* of $\mathcal{P}(\mathcal{S})$. A significant question about polynomial hulls is which hulls have analytic selections.

An obvious necessary condition is that $\mathcal{P}(\mathcal{S})$ in $\mathbf{C}^{1+N}$ be a set whose projection onto the first coordinate is a set containing $\mathbf{D}$. We shall refer to such $\mathcal{P}(\mathcal{S})$ as having *nontrivial fiber over the unit disk.* Also, if the $\mathcal{S}_\theta$ are not connected then it is easy to make up examples where $\mathcal{P}(\mathcal{S})$ has no analytic selection.

Question (Q). Are these conditions sufficient for $\mathcal{P}(\mathcal{S})$ to have an analytic selection?

By giving a highly pathological example (for $N=1$), Wermer [Wr] showed that in general the answer is No. However, when the $\mathcal{S}_\theta$ are nicely behaved the story is different. For $N=1$ Slodkowski [Sl] and independently Wegert [Wg] and Helton–Marshall [HM] showed that the answer is Yes. This article gives a simple very well-behaved $\mathcal{S}$ in $\mathbf{C}^3$ for which the answer to (Q) is No.

Now we write down the $\mathcal{S}$ that provides our example. Write $\mathbf{C}^2$ as $\{z=(x_1,y_1,x_2,y_2)=(z_1,z_2)\}$. Let $\mathcal{C}$ denote the semicircle

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