Tangents to Curves and a Dyadic Parameterization

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1. Introduction

A rectifiable arc A in the complex plane is usually assumed to be parameterized by arc length along A. The geometric properties of A are then described in terms of the arc-length measure on A. A basic result of the theory is the well-known fact that a rectifiable arc has a tangent almost everywhere with respect to arc-length measure. Our purpose in this paper is to characterize those arcs that have a tangent almost everywhere with respect to Hausdorff linear measure. To this end we introduce a dyadic parameterization of curves, applicable to nonrectifiable Jordan arcs, and our characterization is in terms of this parameterization.

There are many ways of defining a tangent at a point (see Section 3, for example). We shall say that a curve has a tangent at a point when the following definition holds.

DEFINITION 1. Suppose that $A=\{A(x); 0\leq x\leq L\}$ is a Jordan arc in the plane. For $0<x_0<L$ we say that the real axis is *tangent* to A at $0=A(x_0)$ if for each $\delta>0$ there exists $r>0$ such that for $z=A(x)$ with $|z|<r$, $x<x_0$ implies that $|\arg z-\pi|<\delta$ and $x>x_0$ implies that $|\arg z|<\delta$, for appropriate choice of the argument. The arc A has tangent line T at arbitrary $z_0=A(x_0)$ if, after a translation of z_0 to 0 and rotation about the origin taking T to the real axis, the real axis is tangent to the transformed curve at 0.

The connection between rectifiability of curves and function theory is the theorem of F. and M. Riesz, which states that for rectifiable Jordan curves, harmonic measure and arc length are mutually absolutely continuous: If f is a conformal mapping of the interior of the rectifiable curve C onto the interior of the unit disk, extended continuously to a homeomorphism of the boundaries, then sets of Lebesgue measure zero on the unit circle correspond to sets of arc-length measure zero on C, and conversely.

In the following, the word "curve" shall always denote a closed Jordan curve; otherwise we use the word "arc".

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