Invariant Subspaces in Bergman Spaces and the Biharmonic Equation

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1. Introduction

Let $\Omega \subset \mathbf{C}$ be a bounded domain in the complex plane. For $0 < p < \infty$, the *Bergman space* $A^p(\Omega)$ consists of all functions f analytic in Ω for which

$$\|f\|_p^p = \int_\Omega |f(z)|^p \, d\sigma < \infty.$$

Here $d\sigma$ denotes the normalized element of area, so that $\|1\|_p = 1$.

In our previous paper [1] we developed a theory of contractive zero-divisors in $A^p = A^p(\mathbf{D})$ for $1 \leq p \leq \infty$, where $\mathbf{D}$ is the unit disk. The general approach through an extremal problem had been introduced by Hedenmalm [3] for the case $p = 2$, but new methods were needed for other values of p. We exploited the positivity of the biharmonic Green function in the disk, and we proved the regularity of the canonical divisors by representing them in terms of the reproducing kernels of certain weighted A^2 spaces.

Our purpose is now to generalize the theory in various directions. It turns out that the arguments in [1], if suitably arranged, actually give a theory of contractive zero-divisors in A^p spaces with $0 < p < 1$. Furthermore, some of the theory applies to arbitrary invariant subspaces (under multiplication by polynomials) and is not restricted to the special invariant subspaces defined by zero-sets. Hedenmalm [3] has already pointed this out for the case $p = 2$. The key to our more general results is an integral formula involving the biharmonic Green function, which has the advantage of circumventing Hedenmalm's boundary-value problem and the consequent need for smooth boundary values. Finally, we show that most of the results extend to Bergman spaces $A^p(\Omega)$ over *simply connected* Jordan domains with analytic boundary. The paper concludes with some special observations and remarks.

2. Background

Again let $A^p(\Omega)$ be the Bergman space over a bounded domain Ω, with $0 < p < \infty$. It is easy to see that even for $p < 1$, each point-evaluation is a bounded

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