Noncollarable Ends of 4-Manifolds: Some Realization Theorems

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A fundamental result in manifold theory is Siebenmann's classification of collarable ends of noncompact n-manifolds, $n \geq 6$ (see [Si]). Quinn [Qu] has extended this result to dimension 5 provided the fundamental group at infinity is a Freedman group. Work by Husch and Price [HP] establishes Siebenmann's theorem for 3-manifolds, provided the Poincaré conjecture is true. Remarkably, Siebenmann's theorem fails in dimension 4. Counterexamples are produced by Kwasik and Schultz in [KS]. These examples arise as quotient spaces of certain free G-actions on $S^3 \times \mathbf{R}$ where G is a finite group of even order. In this note we show that in many cases these exotic ends may be realized rather naturally as subsets of closed 4-manifolds. In particular, we show that if E is a 4-dimensional weak collar with $\pi_1(E) \cong \mathbf{Z}_n$ and ∂E is $\mathbf{Z}$-homology equivalent to $L(n,1)$, then there is a closed 4-manifold Y and a compactum $\Sigma \subset Y$ such that Σ has the shape of a 2-sphere and Σ has a neighborhood N with $N - \Sigma$ homeomorphic to E. Moreover, we may choose Y to be $S^2 \times S^2$ if n is even, and $\mathbf{C}P^2 \# (-\mathbf{C}P^2)$ if n is odd. This (the finite cyclic) case is especially interesting to us because it provides negative answers to some questions raised in [LV2]. One such question asks: If Σ is a globally 1-alg shape 2-sphere in a 4-manifold Y, must the end of $Y - \Sigma$ be collarable?

Another class of Kwasik–Schultz counterexamples to a 4-dimensional collaring theorem contains ends with fundamental group isomorphic to the Poincaré dodecahedral group. We show that these may be realized as complements of cell-like subsets of S^4.

1. Background

All results presented here are topological, as opposed to smooth or PL. Manifolds are permitted to have boundary unless stated otherwise. Homology is with $\mathbf{Z}$-coefficients except where noted to the contrary. Throughout the paper the symbols $\approx$ and $\cong$ represent homeomorphisms and (algebraic) isomorphisms, respectively.

Our primary source for terminology and results involving noncompact 4-manifolds will be [FQ, §11.9]. A similar development can be found in [KS].

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