A Note on Hardy Spaces and Functions of Bounded Mean Oscillation on Domains in $\mathbf{C}^n$

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1. Introduction

It has been considered a part of the folklore for some time that the result of C. Fefferman identifying the dual of $H^1(\mathbf{R}^N)$ as $\mathrm{BMO}(\mathbf{R}^N)$ can be extended (in suitable form) to the unit ball in $\mathbf{C}^n$. In fact the result for the ball appeared *in extenso* in an unpublished version of [CRW]. The main purpose of this note is to give a proof of the theorem in the more general context of strongly pseudoconvex domains in $\mathbf{C}^n$, and in the case of pseudoconvex domains of finite type in $\mathbf{C}^2$.

In X be a Hausdorff space. A *quasimetric* d on X is a continuous function $d: X\times X\to \mathbf{R}^+$ which satisfies the usual requirements for a topological metric except that the triangle inequality is replaced by

$$d(x,z)\le C(d(x,y)+d(y,z)),\quad x,y,z\in X.$$

Let Ω be a smoothly bounded domain in $\mathbf{C}^n$ ($n\ge 2$). We define $\mathcal{H}^1(\Omega)$ to be the usual Hardy space of holomorphic functions on Ω (see [K1]). We may identify it as a closed subspace of $L^1(\partial\Omega)$ by passing to the (almost everywhere) radial limit function $\tilde{f}$ on $\partial\Omega$. Let d be a quasimetric on $\partial\Omega$. Then $\mathrm{BMO}(\partial\Omega)$ can be defined in the usual way, in terms of the quasimetric d and the Lebesgue measure on $\partial\Omega$: the semi-norm on BMO is

$$\|g\|_{\mathrm{BMO}}=\sup_{x,r}\frac{1}{|B(x,r)|}\int_{B(x,r)}|g(t)-g_{B(x,r)}|\,d\sigma(t).$$

Here the balls $B(x,r)$ are defined using the quasimetric, $g_{B(x,r)}$ is the average of g over the ball, $d\sigma$ is $(2n-1)$-dimensional area measure on the boundary of Ω, and $|B(x,r)|=\sigma(B(x,r))$. Of course in practice it is important to select a quasimetric that is compatible with the complex structure.

Now $\mathrm{BMO}A(\Omega)$ denotes the space of holomorphic functions in $\mathcal{H}^1(\Omega)$ whose boundary values are in $\mathrm{BMO}(\partial\Omega)$ with norm $\|f\|_*=\|\tilde{f}\|_1+\|\tilde{f}\|_{\mathrm{BMO}}$. It is easy to prove that $\mathrm{BMO}A(\Omega)$ is a proper closed subspace of $\mathrm{BMO}(\partial\Omega)$.

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