On the Darboux–Picard Theorem in $\mathbf{C}^n$

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I. Introduction

In one complex variable we have the following Darboux–Picard theorem.

Theorem. *Let D be an open disc, and let $f: \bar{D} \to \mathbf{C}$ be continuous and satisfy:*

(i) *f is holomorphic in D, and*
(ii) *f is one-to-one on bD.*

Then f is one-to-one throughout $\bar{D}$ and $f(D)$ is the inside of the Jordan curve $f(bD)$.

For a proof see, for instance, Burckel [1, p. 310].

In this note we will show that the same result still holds if D is sitting in $\mathbf{C}^n$ for $n \geq 2$. But first, a simple example shows that if we map the unit disc in $\mathbf{C}$ into some $\mathbf{C}^n$ with $n > 1$, then in general the conclusion does not hold.

Example. Let U be the unit disc in $\mathbf{C}$. Define $G: \bar{U} \to \mathbf{C}^2$ by

$$z \mapsto (z(z-\tfrac{1}{2})(z+\tfrac{1}{2}), 2(z-\tfrac{1}{2})(z+\tfrac{1}{2})).$$

Then we have $G(\frac{1}{2}) = G(-\frac{1}{2}) = (0,0)$ and G is one-to-one on bU.

Here is our main result.

Main Theorem. *Let $D \subseteq \mathbf{C}^n$, $n \geq 2$, be a bounded domain with bD homeomorphic to S^{2n-1}, and let $f = (f_1, \dots, f_n): \bar{D} \to \mathbf{C}^n$ be a mapping such that $f_j \in H(D) \cap C^0(\bar{D})$ for $j = 1, \dots, n$. Suppose that f is one-to-one on bD; then f is one-to-one throughout $\bar{D}$.*

Some related problems were considered in Globevnik and Stout [2].

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